

Towards softer scales in hot QCD

F. Guerin

Institut Non Lineaire de Nice, 1361 route des Lucioles, 06560 Valbonne, France

Abstract

An effective theory for the soft field modes in hot QCD has been obtained recently by integrating out the field modes of momenta of order T and gT . The mean hard particle distribution obeys a transport equation with a collision term. The scale that comes out for the soft field is $g^2T \ln 1/g$.

n -gluon soft amplitudes are shown to be a sum of tree-diagrams and to obey tree-like Ward identities. These effective amplitudes are used to compute the one-soft-loop contribution to the gluon self-energy when the loop momentum is of order $g^2T \ln 1/g$. It allows to identify a new collision term which, when inserted into the transport equation, takes into account collisions with gluon exchange $\sim g^2T \ln 1/g$. The infrared behaviour of the collision term exhibits a significant change.

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I. INTRODUCTION

The framework of this paper is pure gauge theory in thermal equilibrium. with $g \ll 1$ It has been realized that the near-equilibrium dynamics of high temperature QCD involves a hierarchy of length scales

scale $(T)^{-1}$ inverse of the typical momentum of the plasma particles

scale $(gT)^{-1}$ electric screening length and first scale of collective excitations

scale $(g^2 T \ln 1/g)^{-1}$ damping length of colour excitations

scale $(g^2 T)^{-1}$ non perturbative magnetic fluctuations.

Effective theories can be constructed at a given scale that integrate out the smaller scales. The prototype is the hard thermal loop (HTL) effective theory that integrate out the scale $(T)^{-1}$ [1–3]. To integrate out the hard modes $p \sim T$ means to calculate loop diagrams with external momenta $\ll T$ and internal momenta of order T . It generates effective propagators and effective vertices for the field modes with momenta $\ll T$. To leading order, only one-loop diagrams survive and the result is the hard thermal loop n -point functions. Remarkably, the generating functional of the hard thermal loops is gauge invariant.

Bödeker’s break-through [4] was to realize that one may integrate out the scale $(gT)^{-1}$. The summation now involves an infinite set of multiloop diagrams with loop momenta gT and effective HTL propagators and effective vertices.

One way to construct effective theories for the soft field modes of the plasma is to use classical transport equations. The soft field modes behave classically, this is due to the fact that the momentum scales of interest are small compared to the temperature, so that the modes have large occupation numbers. The resulting set of equations that characterize the effective theory has now been obtained through different approaches [4–8]. This work will follow the approach of Blaizot and Iancu [9,10] to the construction of the effective theories, we summarize it here.

At high temperature, the non-abelian pure gauge theory describes a weakly coupled plasma whose constituents, gluons have a typical momenta $k \sim T$ and a typical distance $(T)^{-1}$. These plasma particles may take part in long-wavelength collective excitations, which are fluctuations in the average density of the plasma particles. These excitations may be induced by a weak external disturbance. The collective excitations at scale x are described by a field $W(x, \mathbf{v})$, a colour matrix in the adjoint representation $W(x, \mathbf{v}) = W^a(x, \mathbf{v}) T^a$, where \mathbf{v} is the direction of propagation of the excitation. Transport equations describe the space-time evolution of those long-wavelength excitations.

The soft fields that one is interested in, are represented by mean fields. In the transport equation, they couple to the collective excitations via a mean-field term $\mathbf{v} \cdot \mathbf{E}^a(x)$ and via the covariant derivative. The soft fields obey Maxwell equations, where the collective excitations act as an induced source

$$D^\nu F_{\mu\nu}(x) = j_\mu^{ext}(x) + j_\mu^{ind}(x) \quad (1.1)$$

where the induced current is the response of the plasma to the initial perturbation $j_\mu^{ext}(x)$

$$j_\mu^{a\ ind}(x) = m_D^2 \int \frac{d\Omega_{\mathbf{v}}}{4\pi} v_\mu W^a(x, \mathbf{v}) \quad (1.2)$$

with

$$m_D^2 = \frac{g^2 N T^2}{3} \quad (1.3)$$

On the space-time scale $(gT)^{-1}$, the transport equation is

$$v.D_x^{ab} W^b(x, \mathbf{v}) = \mathbf{v} \cdot \mathbf{E}^a(x) \quad (1.4)$$

with $v_\mu(v_0 = 1, \mathbf{v})$. The effective theory for the soft field modes is obtained as follows. By solving the transport equation via a Green function, one can express the collective excitation $W(x, \mathbf{v})$ as a functional of the soft fields. Then the induced current, Eq.(1.2), acts as a generating functional for the n -point amplitudes of the soft fields. The resulting amplitudes are the HTL amplitudes.

When one is interested in collective excitations involving colour fluctuations on larger wavelength, one integrates out both scales $(T)^{-1}$ and $(gT)^{-1}$. It turns out that the resulting set of equations is similar. The new element is the inclusion of the effect of collisions and they are dominated by small angle scattering. Indeed a scattering process which hardly changes the momentum of a hard particle, can change its colour charge. The transport equation at larger wavelength $x \gg (gT)^{-1}$ is

$$[v.D_x^{ab} + \hat{C}] W^b(x, \mathbf{v}) = \mathbf{v} \cdot \mathbf{E}^a(x) \quad (1.5)$$

The collision term $\hat{C} = C(\mathbf{v}, \mathbf{v}')$ is non local in \mathbf{v} , but local in x and blind to colour. The gluons $p \sim T$ take part in the collective motion, the gluons $p \sim gT$ are exchanged in the collision process.

The soft fields obey the Maxwell equations (1.1) and the effective theory is obtained from the induced current (1.2). The resulting new feature is the effect of dissipation as a result of collisions. A new scale appears, the colour relaxation time $1/\gamma$ with $\gamma \sim g^2 T \ln 1/g$.

This purely dissipative description is appropriate to the study of the relaxation of a weak initial disturbance at scale x . However colour excitations may also be generated by thermal fluctuations in the plasma. The fluctuation-dissipation theorem relates the two phenomena. In Bödeker's alternative construction of the effective theory (without an external source) [4], the collision term in Eq.(1.5) is accompanied by a Gaussian white noise term, which arises from the thermal fluctuations of initial conditions of the field with momentum $\sim gT$. The noise term keeps the soft modes in thermal equilibrium, it injects energy which compensates for the loss of energy at scale x from the damping term.

One quantity that turns out to be particularly sensitive to the change of scale is the polarization tensor in the magnetic sector. Precisely that quantity enters the collision term of the transport equation in a strategic way. Indeed the interaction rate between two hard particles is dominated by soft momentum transfer $q \sim gT$ where the squared modulus of the resummed gluon propagator enters

$$I \simeq m_D^2 \int_{q_0 < q} \frac{dq_0 d_3 q}{q^4 + (\text{Im}\Pi^t)^2} \frac{1}{q^2}$$

where $1/q^2$ comes from energy-momentum conservation at each vertex

When the scale T has been integrated out, Π^t has the well known HTL expression $\text{Im}\Pi^t =$

$-\pi m_D^2 q_0/q$ for $q_0 \ll q$ so that $I \sim \int dq/q$ leading to the $\ln 1/g$ factor in the scale of the collision term. That expression for Π^t turns out to be valid at the scale gT only. Indeed, when the scale gT is integrated out the resulting Π^t is [10,11] $\text{Im}\Pi^t = -m_D^2 q_0/3\gamma$ for $q_0 \ll q \ll \gamma$ so that the integral I now behaves as $I \simeq \int dq/q^2$. Again this Π^t is valid at the scale $g^2T \ln 1/g$. If one wants to reach the scale g^2T , one has to integrate out the scale $g^2T \ln 1/g$ in order to see how it affects the physics at the scale g^2T .

Arnold and Yaffe [11] have recently attacked the problem of taking the scale $g^2T \ln 1/g$ into account. As they are interested in the colour conductivity, they concentrate on static quantities. They make use of effective theories and have computed the one-loop contribution to $\Pi^l(q_0 = 0, q)$ in an effective theory.

In this work, the one-loop contribution to the polarization tensor $\Pi^{\mu\nu}(q_0, q)$ will be computed for loop momenta $\sim g^2T \ln 1/g$ and external momenta $\sim g^2T$ assuming $\ln 1/g \gg 1$. It is only the first term of an infinite series of multiloop diagrams. However, following a remark made by Blaizot and Iancu [10] for the case of momenta $\sim gT$, it will be possible to identify in the one-loop term, the new collision term that should be added in the transport equation in order to sum the infinite series. This is only a first step towards integrating out the scale $g^2T \ln 1/g$. We will not address the more ambitious question whether integrating out this scale amounts to take into account collisions with exchange gluon $\sim g^2T \ln 1/g$.

The one-soft-loop contribution to the polarization tensor $\Pi^{\mu\nu}(q_0, q)$ comes from a well known pair of diagrams [1,3] whose contribution adds up to give a transverse $\Pi^{\mu\nu}$. One diagram possesses two effective three-gluon vertices and two effective propagators, the other one has an effective four-gluon vertex and an effective propagator. For the case of a loop momentum $\sim g^2T \ln 1/g$, up to now explicit expressions have only been written for the polarization tensor and for the linearized form of the transport equation (1.5) [6,10].

One part of this work will be devoted to write down the explicit form of the needed vertices. In the following, HTL amplitudes refer to the effective amplitudes when the scale $(T)^{-1}$ is integrated out, soft amplitudes refer to the case when the scales $(T)^{-1}$ and $(gT)^{-1}$ are integrated out. The remarkable similarities between the HTL amplitudes and the soft amplitudes will be a recurrent theme in this work:

The n -point amplitudes are obtained from the induced current through the same steps, they obey the same Ward identities. In the one-soft-loop diagram, this similarity will show up again and it will lead to the existence of the new collision operator \hat{C}' which exhibits features similar to the operator \hat{C} . There are also interesting differences, the soft amplitudes contain damping in a way that does not conflict with gauge symmetry.

The transport equation (1.4) has been shown to be gauge invariant, the transport equation (1.5) has been established in the Coulomb gauge, however the collision operator is expected to be gauge-independent [10].

Section 2 is devoted to the polarization tensor $\Pi^{\mu\nu}$. An explicit Lorentz-covariant form is obtained for the case when the scale $(gT)^{-1}$ is integrated out. $\Pi^{\mu\nu}$ appears under a form that allows to interpolate between the scale gT and the scale $g^2T \ln 1/g$. Then it is detailed how the one-loop contribution to the gluon self-energy with loop momenta $g^2T \ln 1/g$ allows to identify the new collision operator, under which assumptions. In Section 3, explicit forms of the effective three-gluon and four-gluon vertices are obtained. The response approach makes use of a Retarded Green function, it leads to the use of the Retarded/Advanced formalism. The n -point Retarded soft amplitudes are obtained from the induced current

and are shown to obey tree-like Ward identities in any leg. The reason is, they are a sum of tree diagrams, the propagator along the tree is the (linearized) one of $W^b(x, \mathbf{v})$, i.e. the soft fields induce the long-wavelength collective excitations. A needed generalization is then made for the four-gluon vertex. In Section 4 the one-soft-loop self-energy diagrams are evaluated in the Retarded/Advanced formalism. The diagram with 3-gluon effective vertices is evaluated first. Loop momenta gT and $g^2T \ln 1/g$ are successively considered. For the case $g^2T \ln 1/g$ one part of the new collision term is identified, its physical interpretation, its scale are examined. Then, the one-loop self-energy diagram with a 4-gluon vertex is treated in a similar way. The total one-loop $\Pi^{\mu\nu}$ is transverse, the total collision operator has a zero mode. The similarities between the two collision operators appear. In Section 5, some properties and consequences of the new collision operator \hat{C}' are studied, in particular its infrared behaviour. Conclusions are in Section 6.

In an Appendix, the explicit expression of $\Pi(q_0, q)$ at the scale $g^2T \ln 1/g$ is written down and its analytical properties in the complex q_0 plane are studied. Another Appendix contains explicit expressions for the eigenvalues of the operator \hat{C}' for the pure magnetic sector's case.

II. THE POLARIZATION TENSOR

Our notations are $K(k_0, \mathbf{k})$, $K^2 = k_0^2 - k^2$

A. The collision-resummed W propagator

The transport equation for the collective W field at a scale $x \gg (gT)^{-1}$ is [4,11,10]

$$(v.D_x + \hat{C})W(x, \mathbf{v}) = \mathbf{v} \cdot \mathbf{E}(x) \quad (2.1)$$

where D_x is the covariant derivative and \hat{C} the collision operator. The linearized equation is in Fourier space

$$(-iv.K + \hat{C}) W(K, \mathbf{v}) = \mathbf{v} \cdot \mathbf{E}(K) \quad (2.2)$$

where $v.K = v_0 k_0 - \mathbf{v} \cdot \mathbf{k}$ and $v(v_0 = 1, \mathbf{v})$ with $\mathbf{v}^2 = 1$, and \hat{C} is an operator in \mathbf{v} space

$$\hat{C} W = \int \frac{d\Omega_{\mathbf{v}'}}{4\pi} C(\mathbf{v}, \mathbf{v}') W(K, \mathbf{v}') \quad (2.3)$$

$C(\mathbf{v}, \mathbf{v}')$ is a real function, symmetric in \mathbf{v} and \mathbf{v}'

$$C(\mathbf{v}, \mathbf{v}') = \gamma \delta_{S_2}(\mathbf{v} - \mathbf{v}') - m_D^2 \frac{g^2 NT}{2} \Phi(\mathbf{v} \cdot \mathbf{v}') \quad (2.4)$$

$$\int \frac{d\Omega_{\mathbf{v}'}}{4\pi} C(\mathbf{v}, \mathbf{v}') = 0 \quad (2.5)$$

and the scale of \hat{C} is set by

$$\gamma \sim g^2 T \ln 1/g \quad (2.6)$$

The explicit form of $\Phi(\mathbf{v}, \mathbf{v}')$ is in Sec.IV B. The eigenvalues of the \hat{C} operator are all positive, as the first term in (2.4) dominates over the second one, except for the eigenvalue zero of \hat{C} , i.e. Eq.(2.5).

The linearized W propagator is the Green function associated with Eq.(2.2) i.e. the inverse operator [11]

$$\hat{G}_R(K, v) = (v.K + i\hat{C})^{-1} \quad (2.7)$$

When \hat{C} is replaced by $\epsilon > 0$, G_R is the retarded Green function associated with the drift equation. Note that this operator in \mathbf{v} space should properly be written

$$\hat{G}_R(K, v) = (v.K \mathcal{I} + iv_0 \hat{C})^{-1} \quad (2.8)$$

where \mathcal{I} is the identity operator in \mathbf{v} space, so that

$$\hat{G}_R(K, v_0, \mathbf{v}) = -\hat{G}_R(K, -v_0, -\mathbf{v}) \quad (2.9)$$

The advanced propagator is

$$\hat{G}_A(K, v) = (v.K - i\hat{C})^{-1} \quad (2.10)$$

and one has

$$\hat{G}_A(K, v) = \hat{G}_R(K, v)^* \quad \hat{G}_A(K, v) = -\hat{G}_R(-K, v) \quad (2.11)$$

Operations in \mathbf{v} space

We adopt the notations of Arnold and Yaffe [11] The measure is $\int d\Omega_{\mathbf{v}}/(4\pi)$

$\langle \dots \rangle_v$ denotes averaging over the direction \mathbf{v}

$$\langle \delta_{S_2}(\mathbf{v} - \mathbf{v}') \rangle_{v'} = 1 = \langle \mathcal{I} \rangle_{v'} \quad (2.12)$$

$$\hat{C} W = \langle C(\mathbf{v}, \mathbf{v}') W(\mathbf{v}') \rangle_{v'} \quad (2.13)$$

$$\langle v_i \hat{C} v_j \rangle = \langle v_i C(\mathbf{v}, \mathbf{v}') v'_j \rangle_{v, v'} \quad (2.14)$$

so that $\langle \dots \rangle$ represent any function independent of \mathbf{v} . As stressed in [11] an important property of \hat{C} is

$$\hat{C} \langle \dots \rangle = 0 \quad \text{or} \quad \langle \hat{C} \dots \rangle = 0 \quad (2.15)$$

from (2.5), with the useful consequence

$$(v.K + i\hat{C})^{-1} v.K \rangle = (v.K + i\hat{C})^{-1} (v.K + i\hat{C}) \rangle = \mathcal{I} \rangle = \langle \dots \rangle \quad (2.16)$$

and a similar one for $(v.K - i\hat{C})^{-1}$

B. The collision-resummed polarisation tensor

Blaizot and Iancu [10] have extracted from the linearized induced current, the form of the polarization tensor that takes into account the full effect of collisions . Defining

$$W(K, \mathbf{v}) = i W^i(K, \mathbf{v}) E^i(K) \quad (2.17)$$

the new functions $W^i(K, \mathbf{v})$ satisfy, from (2.2)

$$(v.K + i\hat{C}) W^i(K, \mathbf{v}) = v^i \quad (2.18)$$

and they obtained (see Eqs.(4.6-4.7) in [10])

$$\Pi^{\mu i} = q_0 m_D^2 \int \frac{d\Omega_{\mathbf{v}}}{4\pi} v^\mu W^i(Q, \mathbf{v}) \quad (2.19)$$

$$\Pi^{\mu 0} = q^i m_D^2 \int \frac{d\Omega_{\mathbf{v}}}{4\pi} v^\mu W^i(Q, \mathbf{v}) \quad (2.20)$$

with $m_D^2 = g^2 N T^2 / 3$. The solution of (2.18) may be written in terms of the retarded inverse operator defined in Eq. (2.7)

$$W_R^i(K, \mathbf{v}) = (v.K + i\hat{C})^{-1} v^i \quad (2.21)$$

Eq.(2.20) become

$$\Pi_R^{\mu i}(Q) = m_D^2 q_0 < v^\mu (v.Q + i\hat{C})^{-1} v^i >_{v,v'} \quad (2.22)$$

$$\Pi_R^{\mu o}(Q) = m_D^2 < v^\mu (v.Q + i\hat{C})^{-1} \mathbf{v} \cdot \mathbf{q} >_{v,v'} \quad (2.23)$$

With $\mathbf{v} \cdot \mathbf{q} = q_0 - v.Q$ and Eq.(2.16), (2.23) becomes

$$\Pi^{\mu 0} = m_D^2 [q_0 < v^\mu (v.Q + i\hat{C})^{-1} >_{v,v'} - < v^\mu \mathcal{I} >_{v,v'}] \quad (2.24)$$

so that $\Pi^{\mu\nu}$ may be written in the compact form

$$\Pi_R^{\mu\nu}(Q) = m_D^2 [q_0 < v^\mu (v.Q + i\hat{C})^{-1} v^\nu >_{v,v'} - g^{\mu 0} g^{\nu 0}] \quad (2.25)$$

(2.25) explicits many properties of $\Pi^{\mu\nu}$. Indeed $\Pi^{\mu\nu}(Q) = \Pi^{\nu\mu}(Q)$ since \hat{C} is symmetric in \mathbf{v} space, and one readily verifies that $Q_\mu \Pi^{\mu\nu} = Q_\nu \Pi^{\mu\nu} = 0$ since, again with Eq.(2.16)

$$Q_\mu \Pi^{\mu\nu} = m_D^2 [q_0 < v^\nu >_{v,v'} - q_0 g^{\nu 0}] = 0 \quad (2.26)$$

Moreover, for momenta $q_0, q \gg g^2 \ln 1/g$, one may replace the operator \hat{C} by $\epsilon > 0$, the W propagator is now diagonal in \mathbf{v} space and one recovers the well-known HTL form for $\Pi^{\mu\nu}$

$$\Pi_R^{\mu\nu}(Q) = m_D^2 [q_0 < v^\mu (v.Q + i\epsilon)^{-1} v^\nu >_{v,v'} - g^{\mu 0} g^{\nu 0}] \quad (2.27)$$

(2.25) interpolates between the collisionless scale $(T)^{-1}$ where the mean field W propagates on a straight line and the scale $(gT)^{-1}$ where collisions cause fluctuations of the direction \mathbf{v} of the W field.

Another aspect of (2.25) follows. An interpretation in terms of tree diagrams may be attached to the first term, i.e. to the Landau damping term. The propagator along the tree is the collision-resummed W propagator. It carries the momentum of the incoming gluon, v^μ is the vertex for a gluon (polarisation μ) attached to the W line. The W propagator is blind to colour. The retarded prescription of the incoming gluon determines the retarded prescription for the W propagator. In Sec.III B the expression (2.25) will be recovered in the framework of the functional derivatives of the induced current.

In Appendix A , an explicit form of $\Pi^{\mu\nu}$ is given, as a continuous fraction, and its analytic properties in the complex q_0 plane are examined.

C. Collision operator and one-loop soft exchange in Π^{ji}

In his quest for an effective theory for the soft field modes $Q \sim g^2 T$, Bödeker's first step was to estimate the contribution from the one-soft-loop diagrams to the polarization tensor $\Pi^{\mu\nu}(Q)$ for loop momentum $K \sim gT$ and $Q \sim g^2 T$. [4,5,12] Those one-loop diagrams involve hard thermal loop effective vertices and propagators.

Then higher loop diagrams with momentum $K \sim gT$ were resummed via a transport equation, so that both scales T and gT were integrated out [4,5]. This transport equation is in terms of the field $W(K, \mathbf{v})$ that describes the collective excitations of the hard particles. Fluctuations in \mathbf{v} are caused by collisions and taken into account via the operator $\hat{C}(\mathbf{v}, \mathbf{v}')$. The linearized transport equation may be written (see Eq.(2.18))

$$(v.K + i\hat{C}) W^i(K, \mathbf{v}) = v^i \quad (2.28)$$

while the integration over the scale T only, lead to the linearized transport equation [9]

$$v.K W^i = v^i \quad (2.29)$$

The complete solution to the transport equation (2.28) involves the inverse operator $(v.K + i\hat{C})^{-1}$

$$W_R^i(K, v) = (v.K + i\hat{C})^{-1} v^i \quad (2.30)$$

As recently stressed by Blaizot and Iancu [10], alternatively one may solve the transport equation (2.28) by iteration. One obtains a series expansion in powers of \hat{C} whose first terms reproduce the expansion in the number of loops with momentum $K \sim gT$. Indeed, writing Eq.(2.28)

$$v.K W^i = v^i - i\hat{C} W^i \quad (2.31)$$

one obtains

$$W_R^{i(0)} = \frac{v^i}{v.K + i\epsilon} \quad , \quad \epsilon > 0 \quad (2.32)$$

$$W_R^{i(1)} = \frac{1}{v.K + i\epsilon} (-i)\hat{C} \frac{v^i}{v.K + i\epsilon} = \frac{1}{v.K + i\epsilon} < (-i)C(\mathbf{v}, \mathbf{v}') \frac{v'^i}{v'.K + i\epsilon} >_{v'} \quad (2.33)$$

Since the polarization tensor is simply related to the W^i field (see Eq. (2.19))

$$\Pi^{ji}(Q) = q_0 m_D^2 < v^j W^i(Q, \mathbf{v}) >_v \quad (2.34)$$

one obtains

$$\Pi^{ji(0)}(Q) = q_0 m_D^2 < v^j \frac{1}{v \cdot Q + i\epsilon} v^i >_v \quad (2.35)$$

$$\begin{aligned} \Pi^{ji(1)}(Q) &= q_0 m_D^2 < \frac{v^j}{v \cdot Q + i\epsilon} (-i) \hat{C} \frac{v^j}{v \cdot Q + i\epsilon} >_{v,v'} \\ &= q_0 m_D^2 < \frac{v^j}{v \cdot Q + i\epsilon} (-i) C(\mathbf{v}, \mathbf{v}') \frac{v'^j}{v' \cdot Q + i\epsilon} >_{v,v'} \end{aligned} \quad (2.36)$$

$\Pi^{ji(0)}$ is the hard thermal loop contribution to the polarization tensor, $\Pi^{ji(1)}$ is the one-loop soft momentum contribution for $K \sim gT$ with $C(\mathbf{v}, \mathbf{v}')$ as in (2.4) [10,12]. The conclusion is, one can identify the collision operator \hat{C} if $\Pi^{ji(1)}$ is known and appears under the form (2.36).

We shall follow this strategy. We want to integrate out the momenta $K \sim g^2 T \ln 1/g$ to see how they affect the dynamics of the soft fields $Q \sim g^2 T$. We shall assume $\ln 1/g \gg 1$ so that the scales are well separated. If one assumes that the summation over the scale $g^2 T \ln 1/g$ can be made via a transport equation which involves a collision operator \hat{C}' , i.e.

$$(v \cdot K + i\hat{C} + i\hat{C}') W^i(K, \mathbf{v}) = v^i \quad (2.37)$$

the full solution will be in terms of an inverse operator

$$W^i = (v \cdot K + i\hat{C} + i\hat{C}')^{-1} v^i \quad (2.38)$$

and the iterative solution will be

$$W^{i(0)} = (v \cdot K + i\hat{C})^{-1} v^i \quad (2.39)$$

$$W^{i(1)} = (v \cdot K + i\hat{C})^{-1} (-i) \hat{C}' (v \cdot K + i\hat{C})^{-1} v^i \quad (2.40)$$

so that

$$\Pi'^{ji(0)}(Q) = q_0 m_D^2 < v^j (v \cdot Q + i\hat{C})^{-1} v^i >_{v,v'} \quad (2.41)$$

$$\Pi'^{ji(1)}(Q) = q_0 m_D^2 < v^j (v \cdot Q + i\hat{C})^{-1} (-i) \hat{C}' (v \cdot Q + i\hat{C})^{-1} v^i >_{all v} \quad (2.42)$$

$\Pi'^{ji(0)}$ is the form found in Eq. (2.25) for the polarization tensor, when the scales T and gT have been integrated out. $\Pi'^{ji(1)}$ will correspond to the one-loop soft momentum exchange $K \sim g^2 T \ln 1/g$, where the one-loop diagrams involve effective vertices and resummed propagators with the scales T and gT integrated out.

The effective vertices will be constructed in Sec.III. The one-loop soft momentum exchange $K \sim g^2 T \ln 1/g$ contribution to the polarization tensor $\Pi'^{ji(1)}$ will be computed in Sec.IV, it will indeed show up as in (2.42). This fact will allow to identify the operator \hat{C}' .

The expression for $C(\mathbf{v}, \mathbf{v}')$ involves by construction an integration over the momentum $K \sim gT$. So does $C'(\mathbf{v}, \mathbf{v}')$ with momentum $K \sim g^2 T \ln 1/g$. There wont be any over-counting in $\hat{C} + \hat{C}'$ in Eqs.(2.37,2.38) if one limits the integration range over the space momentum k

$$\begin{aligned} \mu_2 < k < \mu_1 \quad \text{for } \hat{C} \quad \quad \mu_3 < k < \mu_2 \quad \text{for } \hat{C}' \\ gT < \mu_1 < T \quad , \quad g^2 T \ln 1/g < \mu_2 < gT \quad , \quad g^2 T < \mu_3 < g^2 T \ln 1/g \end{aligned} \quad (2.43)$$

III. THE EFFECTIVES VERTICES

In the response approach, a weak external perturbation is applied to the plasma at some scale x with an adiabatic switching-on at time $-\infty$. The response of the plasma is an induced current. The n -point soft amplitudes are obtained as functional derivatives of this current with respect to the mean field. The response approach involves the Retarded 2-point Green function (the appropriate one to study the response of the plasma to a perturbation which vanishes at time $-\infty$) and it gives out the n -point Retarded amplitudes $V(X_1; X_2, \dots, X_n)$ where X_1^0 is the largest time (see Sec. IIIB). In momentum space, the 2-point Retarded function is the analytical continuation into the upper complex p_0 plane of the Imaginary Time (IT) Green function, i.e. one approaches the real energy axis from above $G(p_0 + i\epsilon) = G_{ret}$. The n -point Retarded amplitude is the analytical continuation $p_i^0 + i\epsilon_i$ of the IT amplitude such that $\epsilon_1 > 0$ and all other $\epsilon_i < 0$. (The constraint $\sum_i p_i^0 = 0$ is imposed on any analytical continuation into the complex energy variables p_i^0 , in particular $\sum_{i=1}^n \epsilon_i = 0$) [13].

Since the known amplitudes will be the Retarded ones, it is natural to use the Retarded/Advanced formalism in computing the two one-soft-loop diagrams that contribute to $\Pi_{\mu\nu}$. Are needed the propagator, the 3-point and 4-point effective vertices. We summarize first the features of this formalism as they will be needed later on, in particular for the 4-point vertex.

A. A summary of the R/A formalism

It is a Real Time formalism where the propagators are the Retarded and Advanced propagators of the zero temperature Field Theory. General properties have been established [14–16]. An external leg l may be of type R or of type A , its incoming energy is $p_l^0 + i\epsilon_l$, $\epsilon_l > 0$ type R , $\epsilon_l < 0$ type A . The n -point amplitudes are defined with all momenta incoming

$$\sum_{l=1}^n p_l^0 = 0 \quad \text{and} \quad \sum_{l=1}^n \epsilon_l = 0$$

Then the only non-zero three-point functions have two legs of type R and one leg of type A , or two legs of type A and one leg of type R . The general properties are, for bosons

$$V(P_{1R}, P_{2A}, P_{3A}) = V^*(P_{1A}, P_{2R}, P_{3R}) \quad (3.1)$$

and for massless bosonic fields in the QED/QCD case [15]

$$V((-P_1)_A, (-P_2)_R, (-P_3)_R) = -V(P_{1R}, P_{2A}, P_{3A}) \quad (3.2)$$

With the same notation, the two-point function, should be written

$$\Pi_R(P) \equiv \Pi(P_R) = \Pi(P_R, (-P)_A) \quad (3.3)$$

and the relations analogous to (3.1,3.2) are

$$\Pi(P_A) = \Pi^*(P_R) \quad (3.4)$$

$$\Pi((-P)_R) = \Pi(P_A) \quad (3.5)$$

Our results will obey those general properties. See (2.25) for $\Pi^{\mu\nu}$.

The n -point Retarded functions that appear in the response approach are $V(P_{1R}, P_{2A}, \dots P_{nA})$ [13], they are related by complex conjugation to the Advanced ones

$$V(P_{1R}, P_{2A}, \dots P_{nA}) = V^*(P_{1A}, P_{2R}, \dots P_{nR}) \quad (3.6)$$

When constructing a diagram, because any n -point vertex is defined with all incoming momenta, a propagator joins an R leg to an A leg since a $P_R(p_0 + i\epsilon, \mathbf{p})$ incoming a vertex comes from another vertex with incoming momentum $(-p_0 - i\epsilon, -\mathbf{p})$ i.e. $(-P)_A$. One can define an ϵ -flow along any tree diagram. This flow is incoming the R legs and outgoing the A legs, it obeys the rules of an electric current since there is no source or sink at a vertex (where $\sum_i \epsilon_i = 0$).

In this R/A formalism, the thermal Bose weight $n(p_0)$ that are associated with the loop momenta in the Imaginary Time formalism, are carried by the vertex in a very specific way, they are attached to the vertices that possess two (or more) legs of type A

$$\Gamma(P_{1R}, P_{2R}, P_{3A}) = V(P_{1R}, P_{2R}, P_{3A}) \quad (3.7)$$

$$\Gamma(P_{1R}, P_{2A}, P_{3A}) = -V(P_{1R}, P_{2A}, P_{3A}) \quad N(P_2, P_3) \quad (3.8)$$

with

$$N(P_2, P_3) = n(p_2^0) + n(p_3^0) + 1 = \frac{n(p_2^0)n(p_3^0)}{n(p_2^0 + p_3^0)} \quad (3.9)$$

$$N(P, -P) = n(p^0) + n(-p^0) + 1 = 0 \quad (3.10)$$

For three legs of type A

$$\Gamma(P_{1R}, P_{2A}, P_{3A}, P_{4A}) = V(P_{1R}, P_{2A}, P_{3A}, P_{4A}) \quad \mathcal{N}(P_2, P_3, P_4) \quad (3.11)$$

with

$$\mathcal{N}(P_2, P_3, P_4) = \frac{n(p_2^0)n(p_3^0)n(p_4^0)}{n(p_2^0 + p_3^0 + p_4^0)} = N(P_2, P_3)N(P_2 + P_3, P_4) \quad (3.12)$$

In order to build one of the pair of one-loop diagrams contributing to the self-energy, a 4-point amplitude with two legs of type R and two legs of type A is needed. Indeed a loop has to be made by joining an R leg to an A leg after the insertion of a propagator. That amplitude possesses a new feature, the prescription $p_i^0 + i\epsilon_i$ on the external legs does not constrain all the energy variables of the amplitude [13,17]. For the case $\epsilon_1 > 0, \epsilon_2 > 0, \epsilon_3 < 0, \epsilon_4 < 0$, ($\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4 = 0$), the energy variable $p_1^0 + p_3^0 = -(p_2^0 + p_4^0)$ may have $\epsilon_1 + \epsilon_3 > 0$ or < 0 . The same for the variable $p_1^0 + p_4^0$. As a result there are, for this case, four distinct analytical continuations of the Imaginary time amplitude, known as Generalized-Retarded amplitudes and they are linked by one relation (see references in [17])

$$\begin{aligned} & V^{(4)}(\epsilon_1 + \epsilon_3 > 0, \epsilon_1 + \epsilon_4 > 0) + V^{(4)}(\epsilon_1 + \epsilon_3 < 0, \epsilon_1 + \epsilon_4 < 0) \\ &= V^{(4)}(\epsilon_1 + \epsilon_3 > 0, \epsilon_1 + \epsilon_4 < 0) + V^{(4)}(\epsilon_1 + \epsilon_3 < 0, \epsilon_1 + \epsilon_4 > 0) \end{aligned} \quad (3.13)$$

The 4-point amplitude RRAA that occurs in the R/A formalism is a specific combination of those analytic continuations [17,15]. The rules are easily stated when the amplitude is a sum of tree diagrams, the only case to be encountered in this work. One splits the tree diagrams into three groups, which depend respectively on $P_1 + P_2, P_1 + P_3, P_1 + P_4$ and

$$\begin{aligned} -\Gamma(P_{1R}, P_{2R}, P_{3A}, P_{4A}) &= F_1((P_1 + P_2)_R)N(P_3, P_4) \\ &+ F_2((P_1 + P_3)_R)N(P_3, P_2 + P_4) + F_2((P_1 + P_3)_A)N(P_4, P_1 + P_3) \\ &+ F_3((P_1 + P_4)_R)N(P_4, P_2 + P_3) + F_3((P_1 + P_4)_A)N(P_3, P_1 + P_4) \end{aligned} \quad (3.14)$$

From Eqs. (3.9, 3.10)

$$N(P_3, P_2 + P_4) + N(P_4, P_1 + P_3) = N(P_3, P_4) = N(P_4, P_2 + P_3) + N(P_3, P_1 + P_4) \quad (3.15)$$

Those weights are compatible with the complex conjugate relation [16]

$$\frac{\Gamma(P_{1R}, P_{2R}, P_{3A}, P_{4A})}{\Gamma^*(P_{1A}, P_{2A}, P_{3R}, P_{4R})} = \frac{N(P_3, P_4)}{N(P_1, P_2)} \quad (3.16)$$

indeed

$$\frac{N(P_3, P_4)}{N(P_1, P_2)} = -\frac{n(p_3^0)n(p_4^0)}{n(-p_1^0)n(-p_2^0)} = \frac{N(P_3, P_2 + P_4)}{N(P_2, P_1 + P_3)} = \frac{N(P_4, P_1 + P_3)}{N(P_1, P_2 + P_4)} = \dots \quad (3.17)$$

as may be shown by writing $N(P_i, P_j) = -N(-P_i, -P_j)$ in each denominator.

B. Soft Amplitudes from the induced current

The soft amplitudes will be obtained as functional derivative of the induced current with respect to the mean field in a way that closely parallels the case of the HTL amplitudes

(see Sec.5 of [9]). One set of Ward Identities satisfied by the amplitudes will follow from the conservation law obeyed by the induced current.

It is convenient to introduce the Retarded Green function associated with the transport equation (1.5) for the $W(X, \mathbf{v})$ field.

$$\begin{aligned} i (v.D_x + \hat{C}) G_{ret}(X, Y; \mathbf{v}, \mathbf{v}') &= \delta^{(4)}(X - Y) \delta_{S_2}(\mathbf{v} - \mathbf{v}') \\ G_{ret}(X, Y; \mathbf{v}, \mathbf{v}') &= 0 \quad \text{for } X_0 < Y_0 \end{aligned} \quad (3.18)$$

with

$$\hat{C} G_{ret}(X, Y; \mathbf{v}, \mathbf{v}') = \int \frac{d\Omega_{\mathbf{v}''}}{4\pi} C(\mathbf{v}, \mathbf{v}'') G_{ret}(X, Y; \mathbf{v}'', \mathbf{v}') \quad (3.19)$$

For the case $\hat{C} = 0$, the solution is known [9] in terms of the parallel transporter along a straight line

$$\begin{aligned} G_{C=0}(X, Y; \mathbf{v}) &= -i \int_0^\infty du \delta^{(4)}(X - Y - vu) U(X, Y) \\ U(X, X - vu) &= P \{ \exp -igu \int_0^1 ds v.A(X - vu(1 - s)) \} \end{aligned} \quad (3.20)$$

where P is the path-ordering operator. An iterative solution to Eq.(3.18) may be written as a power series in \hat{C} in terms of $G_{C=0}$. As a consequence, the following identity, true for $G_{C=0}\delta_{S_2}(\mathbf{v} - \mathbf{v}')$, also holds for $G_{ret}(X, Y; \mathbf{v}, \mathbf{v}')$

$$\frac{\partial G_{ret}(X, Y; \mathbf{v}, \mathbf{v}')}{\partial A_c^\mu(X_1)} = \int \frac{d\Omega_{\mathbf{v}''}}{4\pi} G_{ret}(X, X_1; \mathbf{v}, \mathbf{v}'') gT^c v''_\mu G_{ret}(X_1, Y; \mathbf{v}'', \mathbf{v}') \quad (3.21)$$

where the time arguments satisfy $X^0 \geq X_1^0 \geq Y^0$ and T^c is in the adjoint representation. This identity is of central importance to the structure of the amplitudes.

The solution to the transport equation (1.5) may be written

$$W^a(X, \mathbf{v}) = i \int d^4Y \int \frac{d\Omega_{\mathbf{v}'}}{4\pi} G_{ret}^{ab}(X, Y; \mathbf{v}, \mathbf{v}') \mathbf{v}' \cdot \mathbf{E}^b(Y) \quad (3.22)$$

and from Eq.(1.2) the induced current become

$$j_\mu^a{}^{ind}(X) = im_D^2 \int \frac{d\Omega_{\mathbf{v}}}{4\pi} \int \frac{d\Omega_{\mathbf{v}'}}{4\pi} \int d^4Y v_\mu G_{ret}^{ab}(X, Y; \mathbf{v}, \mathbf{v}') \mathbf{v}' \cdot \mathbf{E}^b(Y) \quad (3.23)$$

Writing [9]

$$\mathbf{v}' \cdot \mathbf{E}(Y) = v'^i F_{0i}(Y) = \partial_{Y_0}(v'^\mu A_\mu) - v' \cdot D_Y A_0(Y) \quad (3.24)$$

Eq.(3.23) may be transformed into a form similar to the case of the HTL amplitudes

$$j_\mu^a{}^{ind}(X) = -m_D^2 g_{0\mu} A_0^a(X) + im_D^2 \int \frac{d\Omega_{\mathbf{v}}}{4\pi} \int \frac{d\Omega_{\mathbf{v}'}}{4\pi} \int d^4Y v_\mu G_{ret}^{ab}(X, Y; \mathbf{v}, \mathbf{v}') \partial_{Y_0}(v' \cdot A^b(Y)) \quad (3.25)$$

Indeed, for the second term on the right handside of (3.24) $v'.D_Y$ is allowed to act on the left on $G_{ret}(X, Y; \mathbf{v}, \mathbf{v}')$; with the use of (3.18), it brings the first term on the right handside of (3.25), the term $G_{ret}(X, Y; \mathbf{v}, \mathbf{v}'')C(\mathbf{v}'', \mathbf{v}')$ does not contribute as the integration over \mathbf{v}' makes use of the property (2.5) of \hat{C} .

The soft one-particle-irreducible amplitudes are then obtained

$$\Pi_{\mu\nu}^{ab}(X, Y) = \frac{\partial j_{\mu}^{a \text{ ind}}(X)}{\partial A_b^{\nu}(Y)} \big|_{A=0} \quad (3.26)$$

and for the $n + 1$ -gluon amplitude

$$g^n V_{\mu\mu_1\cdots\mu_n}^{aa_1\cdots a_n}(X; X_1 \cdots X_n) = \frac{\partial^n}{\partial A_{a_n}^{\mu_n}(X_n) \cdots \partial A_{a_1}^{\mu_1}(X_1)} j_{\mu}^a(X) \big|_{A=0} \quad (3.27)$$

For all these amplitudes, X^0 is the largest time. After differentiation, the Green function that will enter the amplitudes correspond to the case $A = 0$, i. e. to

$$i(v.\partial_X + \hat{C}) G_{ret, A=0}(X, Y; \mathbf{v}, \mathbf{v}') = \delta^{(4)}(X - Y) \delta_{S_2}(\mathbf{v} - \mathbf{v}') \quad (3.28)$$

or in momentum space

$$(v.P + i\hat{C}) G_{ret}(P; \mathbf{v}, \mathbf{v}') = \mathcal{I} \quad \text{i.e.} \quad G_{ret}(P; \mathbf{v}, \mathbf{v}') = (v.P + i\hat{C})^{-1} \quad (3.29)$$

This is the Green function introduced in Sec.II A (see Eq.(2.7)) and called the W propagator.

Moreover, from Eq.(1.1), the induced current obeys the conservation law

$$D^{\mu} j_{\mu}^{a \text{ ind}}(X) = 0 \quad (3.30)$$

Writing $D^{\mu} = \partial^{\mu} \delta^{ab} + g f^{abc} A_c^{\mu}$ and formally

$$j_{\mu}^{a \text{ ind}} = \Pi_{\mu\nu}^{ab} A_b^{\nu} + \frac{g}{2!} V_{\mu\nu\rho}^{aa_2a_3} A_{a_2}^{\nu} A_{a_3}^{\rho} + \frac{g^2}{3!} V_{\mu\nu\rho\sigma}^{aa_2a_3a_4} A_{a_2}^{\nu} A_{a_3}^{\rho} A_{a_4}^{\sigma} + \cdots \quad (3.31)$$

The term in g^{n-2} of (3.30) gives a Ward identity for the X variable, relating the n -point amplitude to $n - 1$ -point amplitudes.

For $\Pi_{\mu\nu}^{ab}(X)$, the functional derivative (3.26) acting on the first term in (3.25) gives

$$-m_D^2 \delta^{ab} g_{0\mu} g_{0\nu} \delta^{(4)}(X - Y)$$

In the second term, the derivative has to act on $\partial_{Y_0} v'.A^b(Y)$. In momentum space, one gets back the expression (2.25) obtained in Sec.II B.

From the properties of the functional derivatives and from the identity (3.21) for the Green function, follow the consequences:

- the $n+1$ -point amplitude is symmetric in the exchange of any pair of the legs $X_1, X_2, \cdots X_n$
- a functional derivative inserts a vertex along a propagator at an intermediate time
- the order in colour space and in \mathbf{v} space follows the time order
- a diagram will correspond to a specific time ordering and there will be a sum over all chronological orderings of $X_1^0, X_2^0, \cdots X_n^0$. X^0 is the largest time, a characteristic feature of

the $n + 1$ -Retarded amplitude

- the form has the minimum number of colour T matrices in the adjoint representation, none for $\Pi_{\mu\nu}$, one for $V_{\mu\nu\rho}$, two for $V_{\mu\nu\rho\sigma}$
- one goes back to the corresponding HTL amplitudes by replacing \hat{C} by $\epsilon\mathcal{I}$ where \mathcal{I} is the identity in \mathbf{v} space.

The Fourier transform of an n -point amplitude is defined

$$(2\pi)^4 \delta(P_1 + P_2 + \dots + P_n) V(P_1, P_2, \dots, P_n) = \int d_4 X_1 \dots d_4 X_n \exp i(P_1 X_1 + \dots + P_n X_n) V(X_1, X_2 \dots X_n) \quad (3.32)$$

all the momenta are incoming the amplitude and the colour indices will be written $a_1 = 1$, $a_2 = 2, \dots$.

All the properties are consequences of the transport equation for the W field. The transport equation (1.5) has been established in the Coulomb gauge, it is expected to be gauge independent [10]. At the HTL level, there is no induced current for the ghost field, i.e. no effective vertices. It is likely to be the same for the soft amplitudes.

C. The 3-gluon vertex

The result of the functional differentiation (3.27) on Eq.(3.25) is in momentum space

$$V_{\mu\nu\rho}^{123}(P_{1R}, P_{2A}, P_{3A}) = im_D^2 \{ \quad (3.33) \\ f^{123} < v_\mu (v.P_1 + i\hat{C})^{-1} v'_\nu (v.(P_1 + P_2) + i\hat{C})^{-1} v''_\rho (-p_3^0) >_{vv'v''} \\ + f^{132} < v_\mu (v.P_1 + i\hat{C})^{-1} v'_\rho (v.(P_1 + P_3) + i\hat{C})^{-1} v''_\nu (-p_2^0) >_{vv'v''} \}$$

$$V_{\mu\nu\rho}^{123}(P_{1R}, P_{2A}, P_{3A}) = im_D^2 f^{123} < v_\mu (v.P_1 + i\hat{C})^{-1} \quad (3.34) \\ [v'_\nu (-v.P_3 + i\hat{C})^{-1} v''_\rho (-p_3^0) - v'_\rho (-v.P_2 + i\hat{C})^{-1} v''_\nu (-p_2^0)] >_{vv'v''}$$

The first term in (3.33) corresponds to the time ordering $X_1^0 > X_2^0 > X_3^0$, the second term to the time ordering $X_1^0 > X_3^0 > X_2^0$. The amplitude is symmetric in the exchange $2 \leftrightarrow 3$ of all indices (momentum, Lorentz, colour), it has no other symmetry. The 3-point amplitude is a Retarded one, on the left handside of (3.33) each momentum has been given an index R or A according to the rule stated in Sec III A. It is worth noticing that the energy p_i^0 that enters the numerator in each term of (3.34) is the one associated with the earliest time, a consequence of the expression (3.25) for the induced current. One may reverse the whole string of operators in \mathbf{v} space. When \hat{C} is replaced by $\epsilon\mathcal{I}$, one recovers Eq.(5.22) of [9].

The Ward identity stemming from the conserved current (3.30) writes

$$p_1^\mu V_{\mu\nu\rho}(P_{1R}, P_{2A}, P_{3A}) = if^{123} [\Pi_{\nu\rho}((P_1 + P_2)_R, P_{3A}) - \Pi_{\nu\rho}((P_1 + P_3)_R, P_{2A})] \quad (3.35)$$

It is obeyed by the expression (3.34) in a straightforward way since from Eq.(2.16)

$$< v.P_1 (v.P_1 + i\hat{C})^{-1} = < \mathcal{I}$$

Moreover there is a Ward identity with respect to P_2 (or P_3). Indeed, in (3.33) when v_ν sits in the middle of a string of operators in \mathbf{v} space, one writes

$$v.P_2 = (v.(P_1 + P_2) + i\hat{C}) - (v.P_1 + i\hat{C})$$

so that

$$(v.P_1 + i\hat{C})^{-1}v.P_2 (v.(P_1 + P_2) + i\hat{C})^{-1} = (v.P_1 + i\hat{C})^{-1} - (v.(P_1 + P_2) + i\hat{C})^{-1} \quad (3.36)$$

When v_ν sits at the end of a string one uses

$$(-v.P_2 + i\hat{C})^{-1}v.P_2 >= -\mathcal{I} >$$

and one gets from (3.33)

$$p_2^\nu V_{\mu\nu\rho}^{123} = if^{123}m_D^2 \{ \quad (3.37)$$

$$- < v_\mu(v.(P_1 + P_2) + i\hat{C})^{-1}v_\rho > (-p_3^0) + < v_\mu(v.P_1 + i\hat{C})^{-1}v_\rho > (-p_3^0 - p_2^0) \}$$

$$p_2^\nu V_{\mu\nu\rho}^{123} = if^{123}[\Pi_{\mu\rho}(P_{1R}, (P_2 + P_3)_A) - \Pi_{\mu\rho}((P_1 + P_2)_R, P_{3A})] \quad (3.38)$$

To summarize, $V_{\mu\nu\rho}^{123}$ obeys tree-like Ward identities with respect to all legs, and this property will generalize to the amplitudes $V(X; X_1, X_2, \dots X_n)$ although the conservation law for the current enforces it only for the X leg (see the 4-point vertex in the next Sec.). The reason is, the expressions are associated with forward-scattering tree diagrams of a collective excitation, i.e. of the W field. If one thinks about the complex conjugate amplitude, where X^0 is the earliest time, one may interpret Eq.(3.33) as follows. The incoming soft gluon P_1 excites a collective excitation of momentum P_1 which propagates, emits successively the soft gluons $(-P_2)$ and $(-P_3)$ and disappears at the last emission. This interpretation is valid at the HTL level as an alternative to the hard-loop interpretation. When the scale gT is integrated out, the new feature is that it is the only surviving one. In contrast, in the one-loop interpretation, there is a symmetry with respect to all legs, so that the leg P_{1R} could sit in the middle (rather than sitting first), there would be a fork in the ϵ -flow (an allowed fact) and one can easily check that the Ward identity (3.35) stemming from the conserved current would not be satisfied when ϵ is replaced by \hat{C} . It is also apparent on the one-loop 4-point function.

D. Four-gluon Vertices

1. The vertex RAAA

For the 4-point amplitude, the result of the functional differentiation is

$$V_{\mu\nu\rho\sigma}^{1234}(P_{1R}, P_{2A}, P_{3A}, P_{4A}) = m_D^2 < v_\mu(v.P_1 + i\hat{C})^{-1} \quad (3.39)$$

$$[f^{12m}f^{34m}v_\nu(v.(P_1 + P_2) + i\hat{C})^{-1}v_\rho(-v.P_4 + i\hat{C})^{-1}v_\sigma(-p_4^0) + \text{5 terms}] >_{all\ v}$$

where +5 terms mean the addition of the permutations $i \leftrightarrow j$ (in all indices) in any pair among the legs 2, 3, 4. The written term corresponds to the time order $X_1^0 > X_2^0 > X_3^0 > X_4^0$, the order in colour indices and Lorentz indices follows the order in time. An interpretation may be given in terms of tree diagrams, the gluons $(-P_2), (-P_3), (-P_4)$ are emitted by the collective excitation induced by the gluon P_1 . This 4-gluon amplitude's form was written in [12] for the case $\hat{C} = 0$, i.e. for the HTL case in the Imaginary Time formalism.

The following tree-like Ward identities are satisfied

$$\begin{aligned} ip_1^\mu V_{\mu\nu\rho\sigma}^{1234}(P_{1R}, P_{2A}, P_{3A}, P_{4A}) = \\ f^{12m} V_{\nu\rho\sigma}^{m34}((P_1 + P_2)_R, P_{3A}, P_{4A}) + f^{13m} V_{\rho\nu\sigma}^{m24}((P_1 + P_3)_R, P_{2A}, P_{4A}) \\ + f^{14m} V_{\sigma\nu\rho}^{m23}((P_1 + P_4)_R, P_{2A}, P_{3A}) \end{aligned} \quad (3.40)$$

a consequence of the conserved current, and for the leg P_4 (or P_3 , or P_2)

$$\begin{aligned} ip_4^\sigma V_{\mu\nu\rho\sigma}^{1234}(P_{1R}, P_{2A}, P_{3A}, P_{4A}) = \\ f^{41m} V_{\mu\nu\rho}^{m23}((P_1 + P_4)_R, P_{2A}, P_{3A}) + f^{42m} V_{\mu\nu\rho}^{1m3}(P_{1R}, (P_2 + P_4)_A, P_{3A}) \\ + f^{43m} V_{\mu\nu\rho}^{12m}(P_{1R}, P_{2A}, (P_3 + P_4)_A) \end{aligned} \quad (3.41)$$

in a way similar to the case of the 3-point vertex and with the use of the Jacobi identity

$$f^{12m} f^{34m} + f^{42m} f^{13m} + f^{32m} f^{41m} = 0 \quad (3.42)$$

It has to be remembered that the 3-point vertex $V_{\mu\nu\rho}^{123}(P_{1R}, P_{2A}, P_{3A})$ as given by (3.34) is symmetric with respect to the two legs A, and that the R leg sits first in \mathbf{v} space.

It is worth emphasizing the origin of the tree-like structure of the n -point Retarded amplitudes. The existence of the covariant derivative is the only source of non-linearity in $A(X)$ because the mean gauge field enters linearly in the induced current in Eq. (3.25), just as in the HTL case.

2. The vertex RRAA

What enters a one-loop self energy diagram is a 4-point function of the type RRAA; indeed a loop has to be made by joining an R leg to an A leg. The 4-point function just obtained is of the type RAAA (and its complex conjugate ARRR). For the amplitudes of the type RRAA, as explained in Sec.III A, a new feature appears. The prescription $\epsilon_1 > 0, \epsilon_2 > 0, \epsilon_3 < 0, \epsilon_4 < 0$ allows for the energy variable $p_1^0 + p_3^0 = -(p_2^0 + p_4^0)$, $\epsilon_1 + \epsilon_3$ to be either > 0 or < 0 , and one has to sum over the two possible analytical continuations of the Imaginary Time amplitude in that variable with appropriate weights (see Eq.(3.14)). The same for the variable $p_1^0 + p_4^0 = -(p_2^0 + p_3^0)$. A consequence is that the Ward identities take forms somewhat different from the ones just written. For example, instead of Eq.(3.40), one has

$$\begin{aligned} ip_1^\mu V_{\mu\nu\rho\sigma}^{1234}(P_{1R}, P_{2R}, P_{3A}, P_{4A}) = f^{12m} V_{\nu\rho\sigma}^{m34}((P_1 + P_2)_R, P_{3A}, P_{4A}) \\ + f^{13m} \left[\frac{N(P_3, P_2 + P_4)}{N(P_3, P_4)} V_{\nu\rho\sigma}^{2m4}(P_{2R}, (P_1 + P_3)_R, P_{4A}) \right. \end{aligned}$$

$$\begin{aligned}
& + \frac{N(P_4, P_1 + P_3)}{N(P_3, P_4)} V_{\nu\rho\sigma}^{2m4}(P_{2R}, (P_1 + P_3)_A, P_{4A})] \\
& + f^{14m} [3 \leftrightarrow 4]
\end{aligned} \tag{3.43}$$

where $[3 \leftrightarrow 4]$ means a term obtained from the factor that multiplies f^{13m} by the exchange of all indices of 3 and 4. $N(P_i, P_j)$ is defined in Eq.(3.9), and from (3.15)

$$N(P_3, P_2 + P_4) + N(P_4, P_1 + P_3) = N(P_3, P_4)$$

In the response approach, the amplitudes RRAA have to be extracted from correlation functions higher than the 2-point ones, i.e. they are beyond the truncation of the Schwinger-Dyson hierarchy made by Blaizot and Iancu [9]. However the tree-like structure found for the RAA...A soft amplitudes very likely extends to all soft amplitudes. We have been able to write down a tree-like amplitude which satisfies all the needed requirements, i.e.

- direct causality flow along each tree
- symmetry with respect to the two legs of type R, and with respect to the two legs of type A
- Ward identity similar to Eq.(3.43) in any leg.

Our procedure follows. At the HTL level, all 4-point amplitudes arise from different analytical continuations of the IT amplitude. At the one-loop level, it has been shown explicitly how the R/A formalism, which introduces a weight on the tree-like vertices RAA, do lead for the RRAA amplitudes to a specific combination of analytical continuations of the IT amplitude [17,15]. We have started from the one-loop HTL 4-point amplitude, written a la Frenkel-Taylor [2], i.e. with a colour factor $\text{Tr}(T_1 T_2 T_3 T_4)$, with appropriate ϵ -flow and weight $N(P_i, P_j)$ for the case RRAA. Summing over 6 permutations, the colour structure has been reduced to the simplest one, i.e. to factors such as $f^{12m} f^{34m}$. Then, using partial fraction expansion, it has been transformed into a direct ϵ -flow along each tree. By construction, this amplitude satisfies all the required properties listed above. Then a form has emerged that generalizes to the case when each ϵ is replaced by the operator \hat{C} in \mathbf{v} space, and satisfies all requirements. It is

$$\begin{aligned}
& V_{\mu\nu\rho\sigma}^{1234}(P_{1R}, P_{2R}, P_{3A}, P_{4A}) N(P_3, P_4) m_D^{-2} = \frac{1}{2} f^{12m} f^{34m} N(P_3, P_4) \\
& < \{ v_\mu (v.P_1 + i\hat{C})^{-1} v_\nu - v_\nu (v.P_2 + i\hat{C})^{-1} v_\mu \} (v.(P_1 + P_2) + i\hat{C})^{-1} \\
& \{ v_\rho (-v.P_4 + i\hat{C})^{-1} v_\sigma (-p_4^0) - v_\sigma (-v.P_3 + i\hat{C})^{-1} v_\rho (-p_3^0) \} >_{all\ v} \\
& + f^{13m} f^{24m} (p_1^0 + p_3^0) \\
& \{ N(P_3, P_2 + P_4) < v_\mu (v.P_1 + i\hat{C})^{-1} v_\rho (v.(P_1 + P_3) + i\hat{C})^{-1} v_\nu (-v.P_4 + i\hat{C})^{-1} v_\sigma >_{all\ v} \\
& - N(P_4, P_1 + P_3) < v_\nu (v.P_2 + i\hat{C})^{-1} v_\sigma (v.(P_2 + P_4) + i\hat{C})^{-1} v_\mu (-v.P_3 + i\hat{C})^{-1} v_\rho >_{all\ v} \} \\
& + f^{14m} f^{23m} (p_1^0 + p_4^0) \{ 3 \leftrightarrow 4 \} + \mathcal{R}
\end{aligned} \tag{3.44}$$

where $\{ 3 \leftrightarrow 4 \}$ means a term obtained from the factor multiplying $f^{13m} f^{24m}$ by the exchange of all indices of 3 and 4, $N(P_i, P_j)$ is defined in (3.9) and \mathcal{R} is another term that depends on the same propagators that the ones in the factor $f^{12m} f^{34m}$. \mathcal{R} is written explicitly in Appendix B, together with some details on the Ward identities. The terms written explicitly in (3.44) may be seen as a naive extension of the RAAA case; in the

factor of $f^{12m} f^{34m}$, X_1^0 and X_2^0 are the later times, X_3^0 and X_4^0 the earlier ones; in the factor of $f^{13m} f^{24m}$, the first term corresponds to the time ordering $X_1^0 > X_3^0 > X_2^0 > X_4^0$, the second one to $X_2^0 > X_4^0 > X_1^0 > X_3^0$.

What is needed for a self-energy diagram is the forward amplitude with two identical colours, for example colour 2 = colour 3 and $P_2 + P_3 = 0 = P_1 + P_4$. The terms $f^{14m} f^{23m}$ disappear. Moreover it will be shown later on that only the term whose factor is $N(P_3, P_2 + P_4)$ contributes to a soft loop.

IV. THE ONE-LOOP SELF-ENERGY DIAGRAMS WITH EFFECTIVE VERTICES

A. A general expression for the diagram with 3-gluon vertices

At the HTL level, the Retarded/Advanced formalism is an alternative to the Imaginary Time formalism. It avoids the analytic continuations that are necessary in the Imaginary Time formalism. It has been used for explicit calculations of the photon self-energy within the HTL framework at the two-loop level by Aurenche and collaborators [18]. We adopt their conventions.

The propagators are the retarded and advanced effective propagators, i.e. with $\epsilon > 0$

$$\Delta(P_R) = \Delta(p_0 + i\epsilon, p) = \frac{i}{(p_0 + i\epsilon)^2 - p^2 - \Pi(p_0 + i\epsilon, p)} \quad (4.1)$$

$$\Delta(P_A) = \Delta(p_0 - i\epsilon, p) = -\Delta(P_R)^* \quad \Delta(P_R) = \Delta((-P)_A) \quad (4.2)$$

from Eqs.(3.4,3.5) for $\Pi(P)$. In covariant gauges, the effective gluon propagator is

$$D^{\mu\nu}(P) = \mathcal{P}_t^{\mu\nu} \Delta^t(P) + \mathcal{P}_l^{\mu\nu} \Delta^l(P) + i\xi \frac{p^\mu p^\nu}{P^4} \quad (4.3)$$

$$\mathcal{P}_t^{ij} = \delta^{ij} - \hat{p}^i \hat{p}^j, \quad \mathcal{P}_t^{\mu 0} = 0 \quad (4.4)$$

$$\mathcal{P}_l^{\mu\nu} = g^{\mu\nu} - \frac{p^\mu p^\nu}{P^2} - \mathcal{P}_t^{\mu\nu} \quad (4.5)$$

$$\Pi^{\mu\nu}(P) = \mathcal{P}_t^{\mu\nu}(P) \Pi^t + \mathcal{P}_l^{\mu\nu}(P) \Pi^l(P) \quad (4.6)$$

In a Coulomb-like gauge

$$D^{\mu\nu}(P) = \mathcal{P}_t^{\mu\nu} \Delta^t(P) + g_{\mu 0} g_{\nu 0} \frac{P^2}{p^2} \Delta^l(P) + i\xi \frac{p^\mu p^\nu}{p^4} \quad (4.7)$$

The three-gluon vertex has been obtained in Sec.III C, i.e. Eq.(3.34). In this section it will be written

$$g V_{\mu\rho\sigma}^{123}(Q_R, P_A, S_A) = i f^{123} g V_{\mu\rho\sigma}(Q_R, P_A, S_A) \quad (4.8)$$

and the i is dropped to be consistent with the convention for the propagator (4.1). As detailed in Sec.III A, the 3-point vertices to be used in the R/A formalism are

$$\Gamma^{\mu\rho\sigma}(Q_A, P_R, R_R) = V^{\mu\rho\sigma}(Q_A, P_R, R_R) \quad (4.9)$$

$$\Gamma^{\mu\rho\sigma}(Q_R, P_A, R_A) = -V^{\mu\rho\sigma}(Q_R, P_A, R_A)(n(p_0) + n(r_0) + 1) \quad (4.10)$$

$V^{\mu\rho\sigma}(Q_R, P_A, R_A)$ is antisymmetric in the exchange of the two legs of type A, it satisfies the general properties Eqs.(3.1,3.2). Two relations will be useful

$$V^{\mu\rho\sigma}((-Q)_A, (-P)_R, (-R)_R) = -V^{\mu\rho\sigma}(Q_R, P_A, R_A) \quad (4.11)$$

$$n(p_0) + 1/2 = -(n(-p_0) + 1/2) \quad (4.12)$$

When one considers the self-energy $\Pi(Q_R)$, the one-loop diagram with 3-gluon vertices is, in the R/A formalism, the sum of the three diagrams drawn on Fig.1(a)(b)(c) (a propagator joins an R leg to an A leg, see Sec.III A). With the definition of momenta of Fig.1(d), leaving out a colour factor δ_{ab}

$$\Pi_{(3g)}^{\mu\nu}(Q_R) = \frac{Ng^2}{2} \int \frac{d_4p}{i(2\pi)^4} \quad (4.13)$$

$$\begin{aligned} & (n(p_0) + 1/2) [D_{\rho\rho'}(P_R) V^{\rho\mu\sigma}(P_R, Q_R, (-S)_A) D_{\sigma\sigma'}(S_R) V^{\rho'\nu\sigma'}((-P)_A, (-Q)_A, S_R)] \\ & -(n(s_0) + 1/2) [D_{\rho\rho'}(P_A) V^{\rho\mu\sigma}(P_A, Q_R, (-S)_R) D_{\sigma\sigma'}(S_A) V^{\rho'\nu\sigma'}((-P)_R, (-Q)_A, S_A)] \\ & -(n(p_0) - n(s_0)) [D_{\rho\rho'}(P_A) V^{\mu\rho\sigma}(Q_R, P_A, (-S)_A) D_{\sigma\sigma'}(S_R) V^{\nu\rho'\sigma'}((-Q)_A, (-P)_R, S_R)] \end{aligned}$$

Several comments have to be made

- From (4.11) the two vertices entering each term are identical, up to a sign
- The prescription R or A on each leg fixes the way one should approach the discontinuity in that variable. Then one writes $S = P + Q$ and one may integrate over p_0 either on the real axis, or in the complex p_0 plane. Although $s_0 = p_0 + q_0$, it make sense to speak of discontinuities in the p_0 or q_0 or s_0 variable, just as at $T = 0$ one considers singularities in the s or t or u channel of the 4-point function.
- The properties in the complex p_0 plane to be used in the subsequent argument are identical whether $(v.P_R)^{-1}$ enters the HTL vertices, or $(v.P + i\hat{C})^{-1}$ enters the soft vertices. Indeed, when $p_0 + i\epsilon$ enters, the meaning is that one should approach the singularities along the real p_0 axis from above. The operator \hat{C} has an infinite tower of positive eigenvalues c_l and a zero eigenvalue c_0 , i.e. $(v.P + i\hat{C})$ vanishes for $p_0 = \mathbf{v} \cdot \mathbf{p} - ic_l$, i.e. those points are in the lower complex p_0 plane or along the real axis.
- The third term of the right handside of Eq.(4.13) corresponds to diagram(c) of Fig.1. In the first and second term, a term $(n(q_0) + 1/2)$ has been dropped in the thermal weight. The reason is: all factors in the bracket of the first (or second) term have singularities in the p_0 complex plane only on one side of the real axis; by closing the integration contour on the other half plane, they are seen to give a vanishing contribution when they are multiplied by $(n(q_0) + 1/2)$. By the same token $1/2$ could be dropped, it has been kept because of the parity property (4.12) [14]. Conversely, $n(p_0)$ has poles at $p_0 = \pm in2\pi T$ all along the imaginary axis.

Thermal weights depending on the external momenta do appear in the R/A formalism. They

give a vanishing contribution at any loop order because they are associated with terms whose singularities all lie on one side of the real axis. It has to be so in order to agree with the Imaginary Time formalism, where thermal weights only depend on loop momenta (see the coth method [19]). This feature will turn out to be a useful one when one considers soft momentum exchange p .

- Eq. (4.13) is a completely general formula and appears in many cases [18,15].

B. The case of a soft exchange around the loop

We want to use Eq.(4.13) to compute a soft momentum exchange around the loop, i.e. $p_0 \ll T$. Now comes Bödeker's argument [12] (translated from Imaginary Time to R/A formalism). For loop momenta $p_0 \ll T$ one may drop the first and second term in Eq.(4.13). Indeed, as said above, in the first term, the term in brackets has singularities only on one side of the real p_0 axis, and if one closes the integration contour on the other side, the first contribution comes from the pole $p_0 = i2\pi T$ of $n(p_0)$ i.e. a hard energy, and it will lead to terms of order q_0/T . The residue of the pole $p_0 = 0$ of $n(p_0)$ is zero as the two terms in (4.13) whose factor is $n(p_0)$ cancel each other at that point ($\Pi(p_0 = 0, p) = 0$, the vertices are identical up to a sign). In what follows, only one of these two terms is kept, the contribution to be obtained will come from a region away from $p_0 = 0$. The same argument applies to the variable s_0 and the second term.

As a consequence, one is left with the third term, and for $p_0 \ll T$ and $q_0 \ll T$, one may approximate the thermal weights i.e. keep only the pole close to the origin and neglect the other ones' contribution. Changing the loop variable from P to K

$$P = K - Q/2 \quad , \quad S = K + Q/2 \quad (4.14)$$

$$n(p_0) - n(s_0) \approx T \left(\frac{1}{p_0} - \frac{1}{s_0} \right) = \frac{Tq_0}{(k_0 - q_0/2)(k_0 + q_0/2)} \quad (4.15)$$

so that a factor q_0 is extracted from the thermal weight. Note that in terms of the set of diagrams drawn on Fig.1, this argument amounts to keep only the diagram where the vertex with two legs of type A is *not* the one with a $(-Q)_A$ leg. It will be a useful feature when one turns to the other case, i.e. one soft loop diagram with a 4-point vertex.

First, momenta $K \sim gT$ around the loop will be considered and the result obtained in [12] will be reproduced, then momenta $K \sim g^2T \ln 1/g$ will be studied. The vertex V is the total vertex, bare + effective. However diagrams containing only bare vertices (gluon + ghost), or containing one bare and one effective vertex have been shown in [12] to give a subleading contribution, mostly because a bare vertex contains no factor $1/v \cdot Q$ (or $(v \cdot Q + i\hat{C})^{-1}$). At the HTL level, there is no effective ghost vertex, and it is likely to be the same for the case $K \sim g^2T \ln 1/g$. We shall not consider the case $\mu = \nu = 0$ in $\Pi^{\mu\nu}$ as our aim is the collision operator.

For momenta $K \sim gT$ the effective vertex is the HTL one, Eqs.(4.8),(3.34) with $P_1 = Q$, $P_2 = -S = -(K + Q/2)$, $P_3 = P = K - Q/2$ and \hat{C} replaced by $\epsilon \mathcal{I}$ ($\epsilon > 0$) where $\mathcal{I} = \delta_{S_2}(\mathbf{v} - \mathbf{v}')$ is the identity in \mathbf{v} space.

$$V^{\mu\sigma\rho}(Q_R, (-S)_A, P_A) = -m_D^2 < \frac{v_1^\mu v_1^\sigma v_1^\rho}{v_1 \cdot Q_R} \left(\frac{(k_0 + q_0/2)}{v_1 \cdot (K + Q/2)_R} - \frac{(k_0 - q_0/2)}{v_1 \cdot (K - Q/2)_A} \right) >_{v_1} \quad (4.16)$$

$$V^{\mu\sigma\rho}(Q_R, (-S)_A, P_A) = -m_D^2 \mathcal{V}^{\mu\rho\sigma}(K, Q) \quad (4.17)$$

and the resummed propagators (4.3,4.6) are those with the HTL $\Pi^{\mu\nu}$, as in Eq.(2.27), so that keeping only the third term in Eq.(4.13). and with (4.14,4.15)

$$\begin{aligned} \Pi_{(3g)}^{\mu\nu}(Q_R) &= q_0 m_D^4 \frac{g^2 N T}{2} \int \frac{d_4 k}{i(2\pi)^4} \frac{1}{(k_0 - q_0/2)(k_0 + q_0/2)} \\ D_{\rho\rho'}((K - Q/2)_A) D_{\sigma\sigma'}((K + Q/2)_R) \mathcal{V}^{\mu\rho\sigma}(K, Q) \mathcal{V}^{\nu\rho'\sigma'}(K, Q) \end{aligned} \quad (4.18)$$

$\mathcal{V}^{\nu\rho'\sigma'}$ is obtained from $\mathcal{V}^{\mu\rho\sigma}$ with the substitutions $\mu \rightarrow \nu, \rho \rightarrow \rho', \sigma \rightarrow \sigma', v_1 \rightarrow v_2$

If one now drops $Q \sim g^2 T$ in front of $K \sim gT$ in (4.18,4.16)

$$k_0 \left(\frac{1}{v_1 \cdot K_R} - \frac{1}{v_1 \cdot K_A} \right) = k_0 \text{disc} \frac{1}{v_1 \cdot K} = -k_0 2\pi i \delta(v_1 \cdot K) \quad (4.19)$$

and

$$\begin{aligned} \Pi_{(3g)}^{\mu\nu}(Q_R) &= q_0 m_D^4 \frac{g^2 N T}{2} \int \frac{d_4 k}{i(2\pi)^4} D_{\rho\rho'}(K_A) D_{\sigma\sigma'}(K_R) \\ &< v_1^\mu \frac{1}{v_1 \cdot Q_R} v_1^\rho v_1^\sigma \text{disc} \frac{1}{v_1 \cdot K} >_{v_1} < v_2^\nu \frac{1}{v_2 \cdot Q_R} v_2^{\rho'} v_2^{\sigma'} \text{disc} \frac{1}{v_2 \cdot K} >_{v_2} \end{aligned} \quad (4.20)$$

The result does not depend on the gauge parameter ξ owing to the δ functions. $\Pi_{(3g)}^{\mu\nu}$ was found via this perturbative approach in [12] in the Imaginary Time formalism.

Comparing (4.20) with (2.36) for the case $\mu = j, \nu = i$, one identifies the contribution from this diagram to the collision operator \hat{C} , or equivalently to Φ defined in Eq.(2.4)

$$\Phi(\mathbf{v}_1, \mathbf{v}_2) = \int \frac{d_4 k}{(2\pi)^4} |v_1 \cdot D(K) \cdot v_2|^2 (2\pi)^2 \delta(v_1 \cdot K) \delta(v_2 \cdot K) \quad (4.21)$$

with

$$v_1 \cdot D(K) \cdot v_2 = (\mathbf{v}_1 \cdot \mathcal{P}_t \cdot \mathbf{v}_2) \Delta^t(K) + \frac{K^2}{k^2} \Delta^l(K)$$

in both covariant and Coulomb gauges. Φ was also found by other methods [4,6–8]. $\Phi(\mathbf{v}_1, \mathbf{v}_2)$ has been interpreted as the collision cross section of two fast particles with soft momentum exchange K ; the δ functions enforce the conservation of energy-momentum at each vertex in the near-forward direction [10]. The integration range for the space momentum k is limited $\mu_2 < k < \mu_1$ (see (2.43)).

Turning now to the case when the exchange momentum is $K \sim g^2 T \ln 1/g$ the gluon propagators and the 3-gluon vertices have to be modified. The polarization tensor is now as in (2.25) and its related vertex as in (4.8),(3.34) with $P_1 = Q, P_2 = -(K + Q/2), P_3 = K - Q/2$. The only change in the expression (4.18) for the one-loop soft momentum exchange $\Pi_{(3g)}^{\mu\nu}$ is in $\mathcal{V}^{\mu\rho\sigma}$ and $\mathcal{V}^{\nu\rho'\sigma'}$

$$\begin{aligned} \mathcal{V}'^{\mu\rho\sigma}(K, Q) = & \langle v_1^\mu (v_1 \cdot Q + i\hat{C})^{-1} \\ & \{ (k_0 + q_0/2) v_1^\rho (v_1 \cdot (K + Q/2) + i\hat{C})^{-1} v_1^\sigma \\ & - (q_0/2 - k_0) v_1^\sigma (v_1 \cdot (Q/2 - K) + i\hat{C})^{-1} v_1^\rho \} \rangle_{v_1, v'_1, v''_1} \end{aligned} \quad (4.22)$$

and $\mathcal{V}'^{\nu\rho'\sigma'}$ is obtained by the substitution $\mu \rightarrow \nu$, $\rho \rightarrow \rho'$, $\sigma \rightarrow \sigma'$ and $v_1, v'_1, v''_1 \rightarrow v_2, v'_2, v''_2$. The Ward identities (3.35, 3.38) imposed on this 3-gluon vertex read

$$Q_\mu \mathcal{V}'^{\mu\rho\sigma} = m_D^{-2} (\Pi^{\rho\sigma}((K + Q/2)_R) - \Pi^{\rho\sigma}((Q/2 - K)_R)) \quad (4.23)$$

$$- (K + Q/2)_\sigma \mathcal{V}'^{\mu\rho\sigma} = m_D^{-2} (\Pi^{\mu\rho}((Q/2 - K)_R) - \Pi^{\mu\rho}(Q_R)) \quad (4.24)$$

$\Pi_{(3g)}^{\mu\nu}$ is in the desired form (2.42). The factors $v_\mu(v \cdot Q + i\hat{C})^{-1} \dots (v \cdot Q + i\hat{C})^{-1} v_\nu$ come from the tree-like structure of the Retarded effective vertices where $v_\mu(v \cdot Q + i\hat{C})^{-1}$ sits first in \mathbf{v} space.

If one now drops $Q \sim g^2 T$ in front of $K \sim g^2 T \ln 1/g$, which is only valid for $\ln 1/g \gg 1$

$$\begin{aligned} \mathcal{V}'^{\mu\rho\sigma}(K, Q) = & k_0 \langle v_1^\mu (v_1 \cdot Q + i\hat{C})^{-1} \\ & \{ v_1^\rho (v_1 \cdot K + i\hat{C})^{-1} v_1^\sigma - v_1^\sigma (v_1 \cdot K - i\hat{C})^{-1} v_1^\rho \} \rangle_{v_1, v'_1, v''_1} \end{aligned} \quad (4.25)$$

In particular, for ρ and σ spacelike

$$\mathcal{V}'^{\mu ij} = k_0 \langle v_1^\mu (v_1 \cdot Q + i\hat{C})^{-1} \{ v_1^i W_R^j(K, \mathbf{v}_1) - v_1^j W_A^i(K, \mathbf{v}_1) \} \rangle_{v_1} \quad (4.26)$$

where the collective field $W_R^i(K, \mathbf{v})$ has been defined in (2.21) and $W_A^i(K, \mathbf{v})$ in a similar way (see (2.10)). The resulting contribution to the polarization tensor is

$$\Pi'_{(3g)}^{\mu\nu}(Q_R) = q_0 m_D^4 \frac{g^2 N T}{2} \int \frac{d_4 k}{i(2\pi)^4} D_{\rho\rho'}(K_A) D_{\sigma\sigma'}(K_R) \frac{1}{k_0^2} \mathcal{V}'^{\mu\rho\sigma}(K, Q) \mathcal{V}'^{\nu\rho'\sigma'}(K, Q) \quad (4.27)$$

with \mathcal{V}' as in (4.25) or (4.26). A consequence of (4.27) is, from (4.23)

$$\begin{aligned} Q_\nu \Pi'_{(3g)}^{\mu\nu} = & q_0 m_D^2 \frac{g^2 N T}{2} \int \frac{d_4 k}{i(2\pi)^4} D_{\rho\rho'}(K_A) D_{\sigma\sigma'}(K_R) \\ & \frac{1}{k_0^2} \mathcal{V}'^{\nu\rho'\sigma'}(K, Q) (\Pi^{\rho'\sigma'}(K_R) - \Pi^{\rho'\sigma'}(K_A)) \end{aligned} \quad (4.28)$$

In Sec.IV D the other one-loop diagram with a 4-gluon vertex will be computed and one will obtain

$$Q_\nu (\Pi'_{(3g)}^{\mu\nu} + \Pi'_{(4g)}^{\mu\nu}) = 0 \quad (4.29)$$

Comparing (4.27) (4.25) with the expression (2.42) for the case $\mu = j$, $\nu = i$, one obtains the collision term arising from the diagram of Fig.1(d). Defining

$$C'(\mathbf{v}_1, \mathbf{v}_2) = -m_D^2 \frac{g^2 N T}{2} ([\Phi'_{(3g)}(\mathbf{v}_1, \mathbf{v}_2) + \Phi'_{(4g)}(\mathbf{v}_1, \mathbf{v}_2)]) \quad (4.30)$$

$$\begin{aligned}
\Phi'_{(3g)}(\mathbf{v}_1, \mathbf{v}_2) &= \int \frac{d_4 k}{(2\pi)^4} D_{\rho\rho'}(K_A) D_{\sigma\sigma'}(K_R) \\
&< v_1^\rho (v_1.K + i\hat{C})^{-1} v_1'^\sigma - v_1^\sigma (v_1.K - i\hat{C})^{-1} v_1'^\rho >_{v_1'} \\
&< v_2^{\rho'} (v_2.K + i\hat{C})^{-1} v_2'^{\sigma'} - v_2^{\sigma'} (v_2.K - i\hat{C})^{-1} v_2'^{\rho'} >_{v_2'}
\end{aligned} \tag{4.31}$$

where the range of integration over the space momentum is $\mu_3 < k < \mu_2$ (see(2.43)). The reality of $\Phi'_{(3g)}$ is a consequence of the Bose symmetry of the two soft gluons in the effective vertices. Note that one may go back to the expression describing the momentum $K \sim gT$ by replacing the operator \hat{C} by $\epsilon\mathcal{I}$, $\epsilon > 0$, then the inverse operator becomes diagonal in \mathbf{v} space, and one gets back $\Phi(\mathbf{v}_1, \mathbf{v}_2)$ as in (4.21).

The constraint $v_1.K = 0 = k_0 - \mathbf{v}_1.\mathbf{k}$ in Eq.(4.21) is essential for the gauge independence of $\Phi_{(3g)}$ in two ways: i) the contraction $k_\rho v_1^\rho$ vanishes ii) $(\mathbf{v}.\mathbf{k})^2 = k_0^2$ is used to get the same result in covariant and Coulomb gauges.

This is no more true for $\Phi'_{(3g)}$. One has

$$\begin{aligned}
k_\rho &< v_1^\rho (v_1.K + i\hat{C})^{-1} v_1'^\sigma - v_1^\sigma (v_1.K - i\hat{C})^{-1} v_1'^\rho >_{v_1'} \\
&= - < i\hat{C} (v_1.K + i\hat{C})^{-1} v_1'^\sigma >_{v_1'}
\end{aligned} \tag{4.32}$$

which is zero when integrated over v_1 or contracted with k_σ . For $k_0, k \ll \gamma$, it reduces to $-v_1^\sigma$. For a longitudinal gluon exchange, the result depends on the gauge parameter ξ and on the gauge (covariant or Coulomb). As the transport equation (1.5) has been established in the strict Coulomb gauge (and expected to be gauge independent), one may wish to stay in this gauge.

An intuitive picture for the collision term $\Phi'_{(3g)}(\mathbf{v}_1, \mathbf{v}_2)$ is obtained if one limits oneself to the dominant exchange of transverse gluons. From (4.3)

$$D^{\rho\rho'}(K) \approx \Delta^t(K) \mathcal{P}_t^{\rho\rho'} = \Delta^t(K) \mathcal{P}_t^{ii'} \tag{4.33}$$

then from (4.26), the collective field $W^i(K, \mathbf{v})$ appears in $\Phi'_{(3g)}$, its transverse part is parallel to v_t^i

$$W^a(K, \mathbf{v}) = i W^i(K, \mathbf{v}) E^i{}^a(K) \tag{4.34}$$

$$W^i(K, \mathbf{v}) = \hat{k}_i W^l + v_t^i W^t \tag{4.35}$$

$$\mathcal{P}_t^{ii'}(k) W^{i'}(K, \mathbf{v}) = \mathcal{P}_t^{ii'} v^{i'} W^t(K, \mathbf{v}) \tag{4.36}$$

so that

$$\begin{aligned}
\Phi'_{(3g)}(\mathbf{v}_1, \mathbf{v}_2) &= \int \frac{d_4 k}{(2\pi)^4} |\Delta^t(K_R)(\mathbf{v}_1.\mathcal{P}_t.\mathbf{v}_2)|^2 \\
&(-)[W_R^t(K, \mathbf{v}_1) - W_A^t(K, \mathbf{v}_1)] [W_R^t(K, \mathbf{v}_2) - W_A^t(K, \mathbf{v}_2)]
\end{aligned} \tag{4.37}$$

$\Phi'_{(3g)}(\mathbf{v}_1, \mathbf{v}_2)$ may be interpreted as the near-forward collision cross section of two collective excitations. The constraint at each vertex is no more a strict particle-like conservation $2\pi\delta(v_1.K)$ as in (4.21), but a smeared one involving the “width” of the W^t field.

A consequence of (4.31) is

$$\int \frac{d\Omega_{\mathbf{v}_2}}{4\pi} \Phi'_{(3g)}(\mathbf{v}_1, \mathbf{v}_2) = \int \frac{d_4 k}{(2\pi)^4} D_{\rho\rho'}(K_A) D_{\sigma\sigma'}(K_R) \frac{1}{k_0 m_D^2} \quad (4.38)$$

$$(\Pi^{\rho'\sigma'}(K_R) - \Pi^{\rho'\sigma'}(K_A)) < v_1^\rho (v_1 \cdot K + i\hat{C})^{-1} v_1^\sigma - v_1^\sigma (v_1 \cdot K - i\hat{C})^{-1} v_1'^\rho >_{v_1'}$$

One notices that the contribution to the damping rate of a hard gluon arising from the range $K \sim g^2 T \ln 1/g$ is obtained from (4.38) if one substitutes to the \mathbf{v}_1' average the quantity $v_1^\rho v_1^\sigma (-2\pi i) \delta(v_1 \cdot K)$. In Sec. IV D, one will find that

$$\int \frac{d\Omega_{\mathbf{v}_2}}{4\pi} [\Phi'_{(3g)}(\mathbf{v}_1, \mathbf{v}_2) + \Phi'_{(4g)}(\mathbf{v}_1, \mathbf{v}_2)] = 0 \quad (4.39)$$

i.e. a relation similar to (2.5) for the case of the \hat{C} operator.

C. A sum rule for the collision operators \hat{C} and \hat{C}'

In order to know the scale of $\Phi'_{(3g)}$, and of \hat{C}' defined in (4.30), one may average over \mathbf{v}_1 and \mathbf{v}_2

$$\begin{aligned} \bar{\Phi} &= \int \frac{d\Omega_{\mathbf{v}_1}}{4\pi} \frac{d\Omega_{\mathbf{v}_2}}{4\pi} \Phi'_{(3g)}(\mathbf{v}_1, \mathbf{v}_2) \\ &= \int \frac{d_4 k}{(2\pi)^4} D_{\rho\rho'}^*(K_R) D_{\sigma\sigma'}(K_R) \\ &\quad [\Pi^{\rho'\sigma'}(K_R) - \Pi^{\rho'\sigma'}(K_A)] [(\Pi^{\rho\sigma}(K_R) - \Pi^{\rho\sigma}(K_A)) \left(\frac{-1}{k_0^2 m_D^4}\right)] \end{aligned} \quad (4.40)$$

This relation, found in the case $K \sim g^2 T \ln 1/g$, also holds in the case $K \sim gT$, i.e. when $\Phi'_{(3g)}(\mathbf{v}_1, \mathbf{v}_2)$ is replaced by $\Phi(\mathbf{v}_1, \mathbf{v}_2)$. Indeed, for $K \sim gT$, $\Pi^{\rho\sigma}(K)$ is as in (2.27) i.e.

$$\frac{i}{k_0 m_D^2} [\Pi^{\rho\sigma}(K_R) - \Pi^{\rho\sigma}(K_A)] = \int \frac{d\Omega_{\mathbf{v}_1}}{4\pi} v_1^\rho v_1^\sigma 2\pi \delta(v_1 \cdot K) \quad (4.41)$$

and (4.41) inserted into (4.40) leads to $\Phi(\mathbf{v}_1, \mathbf{v}_2)$ as in (4.21). $\bar{\Phi}$ is a gauge independent quantity

$$\bar{\Phi} = \int \frac{d_4 k}{(2\pi)^4} [2|\Delta^t|^2 \left[\frac{2\text{Im}\Pi^t}{k_0 m_D^2}\right]^2 + |\Delta^l|^2 \left[\frac{2\text{Im}\Pi^l}{k_0 m_D^2}\right]^2] \quad (4.42)$$

Relation (4.42) allows an easy comparison between the infrared behaviour of the cases $K \sim gT$ and $K \sim g^2 T \ln 1/g$. One restricts oneself to the dominant transverse gluon exchange. For $k < m_D$, $|\Delta^t(K)|^2$ put a strong weight upon the region $k_0 \ll k$ and one may write

$$\begin{aligned} \text{Im}\Pi^t(k_0, k) &\approx k_0 \text{Im}\tilde{\Pi}^t(k_0 = 0, k) \\ |\Delta^t(K_R)|^2 &= \frac{1}{(k_0^2 - k^2 - \text{Re}\Pi^t)^2 + (\text{Im}\Pi^t)^2} \approx \frac{1}{k^4 + k_0^2 (\text{Im}\tilde{\Pi}^t)^2} \end{aligned} \quad (4.43)$$

$$\int \frac{dk_0}{k^4 + k_0^2 (\text{Im}\tilde{\Pi}^t)^2} = \frac{\pi}{k^2 |\text{Im}\tilde{\Pi}^t|} \quad (4.44)$$

so that from (4.42)

$$\bar{\Phi} \approx \int \frac{dk}{4\pi^3} \frac{k^2}{k^2 |\text{Im}\tilde{\Pi}^t|} \frac{\pi}{2} \left(\frac{2\text{Im}\tilde{\Pi}^t}{m_D^2} \right)^2 \quad (4.45)$$

$$\bar{\Phi} \approx \frac{1}{m_D^2} \frac{2}{\pi^2} \int dk \frac{|\text{Im}\tilde{\Pi}^t|}{k_0 m_D^2} \Big|_{k_0=0}(k) \quad (4.46)$$

- For the region $k \sim gT$, from (4.41)

$$\text{Im}\Pi^t = -k_0 m_D^2 \frac{1}{k} \frac{\pi}{4} \left(1 - \frac{k_0^2}{k^2}\right) \theta(k^2 - k_0^2) \approx -m_D^2 \frac{k_0}{k} \frac{\pi}{4} = k_0 \text{Im}\tilde{\Pi}^t \quad (4.47)$$

so that

$$\bar{\Phi}_1 \approx \frac{1}{m_D^2} \frac{1}{2\pi} \int_{\mu_2} \frac{dk}{k} \quad (4.48)$$

the integral has an infrared logarithmic divergence (μ_2, μ_3 have been defined in (2.43)) and

$$\gamma = m_D^2 \frac{g^2 NT}{2} \quad \bar{\Phi} \approx \frac{g^2 NT}{4\pi} \ln \frac{m_D}{\mu_2} \quad (4.49)$$

- For the region $k \sim g^2 T \ln 1/g$, from Eq.(A7) in Appendix A

$$\Pi^t(k_0, k)m = \frac{m_D^2 k_0}{3} \Sigma_1(k_0, k) \quad (4.50)$$

where Σ_1 is one matrix element of $(v.K + i\hat{C})^{-1}$

$$\bar{\Phi}_2 \approx \frac{1}{m_D^2} \frac{2}{3\pi^2} \int_{\mu_3}^{\mu_2} dk |\text{Im}\Sigma_1(k_0 = 0, k)| \quad (4.51)$$

From (A5,A1) one sees that the scale of Σ_1 is γ and for $k \ll \gamma$

$$\Sigma_1(k_0 = 0, k \ll \gamma) = \frac{-i}{\gamma - \delta_1} \quad \text{and} \quad \text{Im}\tilde{\Pi}^t(0, k) = -\frac{m_D^2}{3} \frac{1}{\gamma - \delta_1} \quad (4.52)$$

so that $\bar{\Phi}_2$ has no infrared divergence. In fact it is unlikely that the main contribution to $\bar{\Phi}_2$ comes from the momenta $k < \gamma/3$. Indeed, as discussed in Appendix A, there is an imaginary part of Π^t in the range $|k_0| < k$ analogous to (4.47) i.e. to the Landau-damping term. However it only exists for $k > \gamma/3$.

Eq.(4.51) sets the scale of the operator \hat{C}' as $g^2 NT$ (see Eq.(4.30)) with a weak dependence on both cutoff.

One may look at the contribution from the momenta $k \sim g^2 T \ln 1/g$ (i.e. in Eq.(4.45), $\text{Im}\tilde{\Pi}^t$ in the denominator is as in (4.52)) to two other quantities

- the damping rate of a hard point-like gluon. Then in $\bar{\Phi}_2$ as in (4.45), one $\text{Im}\tilde{\Pi}^t$ in the numerator is as in (4.52) and the other one as in (4.47), and $\bar{\Phi}_2$ has an infrared log divergence.
- the interaction rate of two point-like hard gluons. Then in $\bar{\Phi}_2$ both $\text{Im}\tilde{\Pi}^t$ in the numerator are as in (4.47), and $\bar{\Phi}_2$ has a linear divergence, as stated in the introduction.

D. The diagram with a 4-gluon effective vertex

The contribution to $\Pi_{\mu\nu}(Q)$ of the diagram drawn on Fig.2(a) is considered. One treats simultaneously both cases i.e. when the momentum k running around the loop is i) $k \sim gT$ (where \hat{C} is replaced by $\epsilon\mathcal{I}$) ii) $k \sim g^2T \ln 1/g$.

As explained in Sec.III A, one has to join the leg P_{3A} to the leg P_{2R} in the 4-gluon vertex

$$ig^2(-) N(P_3, P_4) V_{\mu\sigma\rho\nu}^{1234}(P_{1R}, P_{2R}, P_{3A}, P_{4A})$$

given in Eq.(3.44) with

$$P_3 = -P_2 = K \quad , \quad P_1 = -P_4 = Q \quad (4.53)$$

(Note that the Lorentz indices σ and ν have been exchanged compared to (3.44)). With colour 2 = colour 3, the term $f^{23m}f^{14m}$ disappears in (3.44), all the other colour factors are identical and $\sum_{2,m} f^{12m}f^{24m} = -\delta^{14} N$. Leaving out the colour factor δ^{14}

$$\begin{aligned} \Pi'_{(4g)}{}^{\mu\nu}(Q_R) &= ig^2 N m_D^2 \int \frac{d_4 k}{i(2\pi)^4} D_{\rho\sigma}(P_{2R} = (-K)_R) N(P_3 = K, P_4 = -Q) \\ V_{\mu\sigma\rho\nu}^{1234}(P_{1R} = Q_R, P_{2R} = (-K)_R, P_{3A} = K_A, P_{4A} = (-Q)_A) \end{aligned} \quad (4.54)$$

with

$$N(P_i, P_j) = n(p_i^0) + n(p_j^0) + 1 = n(p_i^0) - n(-p_j^0) \quad (4.55)$$

One may symmetrize the integrant by adding the term obtained in the change of variables $K \rightarrow -K$: $F'(K) = (F(K) + F(-K))/2$.

We now proceed in a way similar to the 3-gluon vertices' case (see Sec.IV B) in order to kill most of the terms appearing in Eqs.(3.44),(B1) :

(i) All the terms whose thermal weight depend on the external momentum Q (i.e. $n(p_4^0) = n(-q_0)$) give a vanishing contribution. In the complex k_0 plane, they have all their singularities on the same side of the real axis, and one may close the contour in the other half plane. Indeed the factor $D_{\rho\sigma}((-K)_R) = D_{\rho\sigma}(-k_0 + i\epsilon, -\mathbf{k})$ is multiplied for example by

$$\begin{aligned} &< v_\sigma(v.P_2 + i\hat{C})^{-1} v_\mu(v.(P_1 + P_2) + i\hat{C})^{-1} v_\nu(-v.P_3 + i\hat{C})^{-1} v_\rho > = \\ &< v_\sigma(-v.K + i\hat{C})^{-1} v_\mu(v.(Q - K) + i\hat{C})^{-1} v_\nu(-v.K + i\hat{C})^{-1} v_\rho > \end{aligned}$$

(ii) For a soft momentum running around the loop $k_0 \ll T$, one may drop all terms whose factor is $N(P_4, P_i) = n(p_i^0) - n(q_0)$, ($P_i = P_3 = K$ or $P_i = P_1 + P_3 = Q + K$), because the first contribution will come from the pole $p_i^0 = 2\pi T$ and it will lead to terms of order q_0/T .

There is no singularity when $p_i^0 \rightarrow 0$ since

- the factor $(p_1^0 + p_3^0)N(P_4, P_1 + P_3)$ is regular as $p_1^0 + p_3^0 \rightarrow 0$

- for $p_3^0 = k_0 \rightarrow 0$, most of the terms whose factor is $n(p_3^0)$ cancel, there remain two terms

$$\dots v_\rho[(v.Q - \mathbf{v.k} + i\hat{C})^{-1} + (v.Q + \mathbf{v.k} + i\hat{C})^{-1}] v_\sigma \dots \quad q_0 n(k_0) D_{\rho\sigma}(k_0 = 0, -\mathbf{k})$$

which cancel with the term obtained in the change of variable $K \rightarrow -K$.

(iii) The only surviving term in Eqs.(3.44),(B1) has singularities on both sides of the real k_0 axis, it is

$$\begin{aligned}
& < v_\mu(v.P_1 + i\hat{C})^{-1}v_\rho(v.(P_1 + P_3) + i\hat{C})^{-1}v_\sigma(-v.P_4 + i\hat{C})^{-1}v_\nu > D_{\rho\sigma}((-K)_R) = \\
& v_\mu(v.Q + i\hat{C})^{-1}v_\rho(v.(Q + K) + i\hat{C})^{-1}v_\sigma(v.Q + i\hat{C})^{-1}v_\nu > D_{\rho\sigma}(-K + i\epsilon)
\end{aligned}$$

with a weight

$$\begin{aligned}
& N(P_3, P_2 + P_4)(p_1^0 + p_3^0) = (n(p_3^0) - n(p_1^0 + p_3^0))(p_1^0 + p_3^0) \\
& \approx \left(\frac{T}{k_0} - \frac{T}{k_0 + q_0}\right)(k_0 + q_0) = \frac{Tq_0}{k_0}
\end{aligned} \tag{4.56}$$

This term is associated with the diagram drawn on Fig.2(b). One adds the term obtained in the substitution $K \rightarrow -K$, i.e.

$$\ldots v_\rho(v.(Q - K) + i\hat{C})^{-1}v_\sigma \ldots D_{\rho\sigma}((K)_R = (-K)_A) \left(-\frac{Tq_0}{k_0}\right)$$

and one may complete with irrelevant terms (singularities in k_0 on same side) to obtain

$$\begin{aligned}
\Pi'_{(4g)}{}^{\mu\nu}(Q) &= iq_0 \frac{g^2 NT}{2} m_D^2 \int \frac{d_4 k}{i(2\pi)^4} \frac{1}{k_0} [D_{\rho\sigma}((-K)_R) - D_{\rho\sigma}(K_R)] \\
&< v^\mu(v.Q + i\hat{C})^{-1}[v^\rho(v.(K + Q) + i\hat{C})^{-1}v^\sigma \\
&+ v^\sigma(v.(Q - K) + i\hat{C})^{-1}v^\rho](v.Q + i\hat{C})^{-1}v^\nu >
\end{aligned} \tag{4.57}$$

Neglecting Q in front of K , the resulting $\Pi'_{(4g)}{}^{\mu\nu}$ is

$$\begin{aligned}
\Pi'_{(4g)}{}^{\mu\nu}(Q) &= -iq_0 \frac{g^2 NT}{2} m_D^2 \int \frac{d_4 k}{i(2\pi)^4} \frac{1}{k_0} [D_{\rho\sigma}(K_R) - D_{\rho\sigma}(K_A)] \\
&< v^\mu(v.Q + i\hat{C})^{-1}[v^\rho(v.K + i\hat{C})^{-1}v^\sigma - v^\sigma(v.K - i\hat{C})^{-1}v^\rho](v.Q + i\hat{C})^{-1}v^\nu >
\end{aligned} \tag{4.58}$$

A consequence is

$$\begin{aligned}
Q_\nu \Pi'_{(4g)}{}^{\mu\nu}(Q) &= -iq_0 \frac{g^2 NT}{2} m_D^2 \int \frac{d_4 k}{i(2\pi)^4} \frac{1}{k_0} [D_{\rho\sigma}(K_R) - D_{\rho\sigma}(K_A)] \\
&< v^\mu(v.Q + i\hat{C})^{-1}[v^\rho(v.K + i\hat{C})^{-1}v^\sigma - v^\sigma(v.K - i\hat{C})^{-1}v^\rho] >
\end{aligned} \tag{4.59}$$

where Eq.(2.16) has been used. One sees that the factor that depends on v in (4.59) is $\mathcal{V}'^{\mu\nu\rho}(K, Q) / k_0$ as in Eq.(4.25). From (4.3,4.6)

$$D_{\rho\sigma}(K_R) - D_{\rho\sigma}(K_A) = -i D_{\rho\rho'}(K_A) D_{\sigma\sigma'}(K_R) (\Pi^{\rho'\sigma'}(K_R) - \Pi^{\rho'\sigma'}(K_A)) \tag{4.60}$$

so that comparing with (4.28) one obtains

$$Q_\mu \Pi'_{(4g)}{}^{\mu\nu} = -Q_\mu \Pi'_{(3g)}{}^{\mu\nu} \tag{4.61}$$

Moreover, comparing $\Pi'_{(4g)}{}^{\mu\nu}(Q)$ as in (4.58) with (2.42) for the case $\mu = j$, $\nu = i$ one extracts a collision term (see (4.30))

$$\begin{aligned} \Phi'_{(4g)}(\mathbf{v}, \mathbf{v}') &= -i \int \frac{d_4 k}{(2\pi)^4} \frac{1}{k_0 m_D^2} [D_{\rho\sigma}(K_R) - D_{\rho\sigma}(K_A)] \\ &[v^\rho (v.K + i\hat{C})_{v,v'}^{-1} v'^\sigma - v^\sigma (v.K - i\hat{C})_{v,v'}^{-1} v'^\rho] \end{aligned} \quad (4.62)$$

$\Phi'_{(4g)}$ is an operator. When applied on a function $>$ that does not depend on \mathbf{v}'

$$\hat{\Phi}'_{(4g)} > = \int \frac{d\Omega_{\mathbf{v}'_1}}{4\pi} \Phi'_{(4g)}(\mathbf{v}_1, \mathbf{v}'_1) = - \int \frac{d\Omega_{\mathbf{v}_2}}{4\pi} \Phi'_{(3g)}(\mathbf{v}_1, \mathbf{v}_2) \quad (4.63)$$

from (4.38) and (4.60), i.e. the collision operator has a zero mode

$$\hat{C}' > = \hat{\Phi}'_{(4g)} > + \hat{\Phi}'_{(3g)} > = 0 \quad (4.64)$$

$\Phi'_{(4g)}$ does not depend on the gauge parameter ξ , but the longitudinal gluon exchange does depend on the gauge (covariant or Coulomb) since $k_0 \neq \mathbf{v} \cdot \mathbf{k}$. If one restrict oneself to transverse gluon exchange, (4.62) may be written

$$\begin{aligned} \Phi'_{(4g)}(\mathbf{v}, \mathbf{v}') &= -i \int \frac{d_4 k}{(2\pi)^4} \frac{1}{k_0 m_D^2} [\Delta^t(K_R) - \Delta^t(K_A)] \\ &v_t^i [(v.K + i\hat{C})^{-1} - (v.K - i\hat{C})^{-1}]_{vv'} v'^i_t \end{aligned} \quad (4.65)$$

To go back to the expression that corresponds to momentum exchange $K \sim gT$, one replaces \hat{C} by $\epsilon\mathcal{I}$ in (4.62) and one gets back the local term in \hat{C} (see (2.4)). Indeed, the identity operator in \mathbf{v} space is $\mathcal{I} = \delta_{S_2}(\mathbf{v} - \mathbf{v}')$, and one recognizes in (4.60,4.62) the hard gluon damping rate γ if \hat{C} is replaced by $\epsilon\mathcal{I}$. $\Phi'_{(4g)}$ comes from the self-energy diagram drawn on Fig.2(b) involving a collective-excitation propagator and a soft-gluon propagator, it is interpreted as the damping of a collective excitation (at a scale larger than $k^{-1} \sim (g^2 T \ln 1/g)^{-1}$), its scale is $\bar{\Phi}_2$ discussed in Sec.IV C.

V. PROPERTIES OF THE COLLISION OPERATOR \hat{C}'

As a result of Sec.IV, the collision operator \hat{C}' that takes into account the collisions with exchanged gluon $k \sim g^2 T \ln 1/g$ is

$$\hat{C}' = C'(\mathbf{v}, \mathbf{v}') = -m_D^2 \frac{g^2 N T}{2} (\Phi'_{(3g)}(\mathbf{v}, \mathbf{v}') + \Phi'_{(4g)}(\mathbf{v}, \mathbf{v}')) \quad (5.1)$$

with $\Phi'_{(3g)}(\mathbf{v}, \mathbf{v}')$ as in (4.31), or (4.37) for transverse gluon exchange, $\Phi'_{(4g)}(\mathbf{v}, \mathbf{v}')$ as in (4.62), or (4.65). Both are functions of \mathbf{v}, \mathbf{v}' as there is no other vector available.

In this section one restricts oneself to the dominant transverse gluon exchange

$$\begin{aligned} \Phi'_{(3g)}(\mathbf{v}_1, \mathbf{v}_2) &= - \int \frac{d_4 k}{(2\pi)^4} |\Delta^t(K_R)|^2 \\ &< v_{1i}^t (v_1.K + i\hat{C})^{-1} v_{1j}^{t'} - v_{1j}^t (v_1.K - i\hat{C})^{-1} v_{1i}^{t'} >_{v'_1} \\ &< v_{2i}^t (v_2.K + i\hat{C})^{-1} v_{2j}^{t'} - v_{2j}^t (v_2.K - i\hat{C})^{-1} v_{2i}^{t'} >_{v'_2} \end{aligned} \quad (5.2)$$

$$\Phi'_{(4g)}(\mathbf{v}, \mathbf{v}') = -i \int \frac{d_4 k}{(2\pi)^4} \frac{1}{k_0 m_D^2} [\Delta^t(K_R) - \Delta^t(K_A)] \quad (5.3)$$

$$v_t^i [(v.K + i\hat{C})^{-1} - (v.K - i\hat{C})^{-1}] v_t'^i$$

where $v_t^i = v^i - \hat{k}^i \mathbf{v} \cdot \hat{\mathbf{k}}$

The operator $C'(\mathbf{v}, \mathbf{v}')$ shares many features of the operator $C(\mathbf{v}, \mathbf{v}')$:

- it commutes with the rotations in \mathbf{v} space, its eigenvalues are

$$c'_l = \langle P_l(\mathbf{v}, \mathbf{v}') C'(\mathbf{v}, \mathbf{v}') \rangle_{v, v'} \quad (5.4)$$

and its eigenvectors are $Y_l^m(\mathbf{v})$,

- \hat{C}' is written as an integral over the soft momenta k of the exchanged gluon in the near-forward collision of two collective excitations of direction \mathbf{v} and \mathbf{v}' , ($k \sim gT$ for \hat{C} , $k \sim g^2 T \ln 1/g$ for \hat{C}'). All the dependance on \mathbf{v} and \mathbf{v}' is in the effective vertices, i.e. in the (smeared) energy-momentum conservation at the vertex

- the soft gluon exchange $k \sim g^2 T \ln 1/g$ put a strong weight upon the region $k_0 \leq (k^2/T) \ln 1/g$ and one can set $k_0 = 0$ in the effective vertices whose scale is $k_0 \sim k \sim g^2 T \ln 1/g$ (with one exception). (For \hat{C} one also sets $k_0 = 0$ to leading order in the vertices). The resulting scale for c'_l is $g^2 NT$, with a weak dependance on both cutoff $\mu_3 < k < \mu_2$.

- the collision operator is made of two terms. One term comes from a self-energy diagram, in fact it is the damping of the collective excitation at a scale larger than $1/k$. It dominates the spectrum of the operator, except for the $l = 0$ eigenvalue where both terms cancel each other, giving rise to a zero eigenvalue. The contribution from the other term vanishes for $l = 1, 3, 5 \dots$ to leading order.

The operators \hat{C} and \hat{C}' also show some differences

- the infrared behaviour of the eigenvalues differ. For a transverse gluon exchange, the c_l have a logarithmic infrared divergence that shows up in both terms of the collision operator. The c'_l are infrared finite with one exception: the $l = 1$ eigenvalue has a linear infrared divergence (see underneath).

- the operator \hat{C} is expected to be gauge independent. As seen in Sec.IV B, \hat{C} takes an identical form in covariant and Coulomb gauges. For \hat{C}' , the longitudinal soft gluon exchange is very likely gauge dependent, apparently a consequence of the finite lifetime of the collective excitations at a smaller scale.

The collision operators \hat{C} and \hat{C}' are so similar that it is easy to take into account both types of collisions, i.e. when the exchanged gluon has $k \sim gT$ and when $k \sim g^2 T \ln 1/g$. One just has to substitute the operator $\hat{C} + \hat{C}'$ to the operator \hat{C} in all the relations written in Sec.II and Sec.III. The transport equation for the W field is now

$$(v.D + i\hat{C} + i\hat{C}') W = \mathbf{v} \cdot \mathbf{E}$$

and from it, one deduces the induced current, then the softer amplitudes. These softer amplitudes are tree-like and they obey tree-like Ward identities.

This approach, via the polarization tensor, proposes an interpretation of \hat{C} and of \hat{C}' somewhat different from the loss and gain interpretation that results from the transport equation via the Schwinger-Dyson approach [6,10]. In that alternative approach, the Bose symmetry of the effective vertices with respect to two legs of the same type is the dominant

constraint, as a consequence the colour factors differ from the ones that appear in the Schwinger-Dyson case.

A. The eigenvalues of the operator \hat{C}'

In Appendix C, matrix elements such as

$$M_{(4g)}^{(l)}(K) = \langle P_l(\mathbf{v} \cdot \mathbf{v}') \ v_i^t [(v \cdot K + i\hat{C})^{-1} - (v \cdot K - i\hat{C})^{-1}] \ v_i^t \rangle_{v,v'} \quad (5.5)$$

are expressed in terms of matrix elements of $(v \cdot K \pm i\hat{C})^{-1}$, which are themselves continued fractions, functions of k_0, k and of all the eigenvalues c_l of the operator \hat{C} .

The case of the eigenvalue $l = 0$ of \hat{C}' has been treated in Sec.IV C. Many encountered features are valid for all c'_l . For $l = 0$, the contributions of both terms $\Phi'_{(3g)}$ and $\Phi'_{(4g)}$ are equal and opposite, it has been called $\bar{\Phi}_2$. $\bar{\Phi}_2$ has been expressed in terms of the imaginary part of the $l = 1, m = 1$ eigenvalue of $(v \cdot K + i\hat{C})^{-1}$ which is finite as $k \rightarrow 0$. One consequence is that the integral over k in (5.2) and (5.3) is infrared finite (see Eq.(4.51)).

One can see that there is no infrared divergence in the contribution of $\Phi'_{(4g)}$ to c'_l in the following way. The scale of the matrix elements of $(v \cdot K + i\hat{C})$ is the scale of \hat{C} , i.e. $\gamma \sim g^2 T \ln 1/g$. For $k \ll \gamma$, the matrix elements of $(v \cdot K + i\hat{C})^{-1} \approx (k_0 + i\hat{C})^{-1}$ do not depend on $m = \mathbf{l} \cdot \hat{\mathbf{k}}$, but only on l , they are the inverse of $\langle P_l(\mathbf{v} \cdot \mathbf{v}') (k_0 + i\hat{C}) \rangle_{v,v'} = k_0 + ic_l$. In Eq.(5.5), there is no dependance on m for $k \ll \gamma$ and one may write

$$\begin{aligned} \mathbf{v}_t \cdot \mathbf{v}'_t \ P_l(\mathbf{v} \cdot \mathbf{v}') &= \frac{2}{3} \ \mathbf{v} \cdot \mathbf{v}' \ P_l(\mathbf{v} \cdot \mathbf{v}') \\ &= \frac{2}{3} \frac{1}{2l+1} [(l+1) P_{l+1}(\mathbf{v} \cdot \mathbf{v}') + l P_{l-1}(\mathbf{v} \cdot \mathbf{v}')] \end{aligned} \quad (5.6)$$

i.e. for $k \ll \gamma$, Eq. (5.5) becomes

$$M_{(4g)}^{(l)}(k_0) = \frac{2}{3} \frac{1}{2l+1} \left[\frac{l+1}{k_0 + ic_{l+1}} + \frac{l}{k_0 + ic_{l-1}} - \text{c.c.} \right] \quad (5.7)$$

where c.c. means complex conjugate. One can set $k_0 = 0$ in $M_{(4g)}^{(l)}(k_0)$ and one concludes that, just as for the case $l = 0$, the integral over k has no infrared divergence as $k \rightarrow 0$. And the region $k \ll \gamma$ of the integrant does not depend on l for large l since $c_l = \gamma - \delta_l$ with $\delta_l \sim \gamma/l$. There is one exception: the case $l = 1$ for c'_l , where one sees from (5.7) that the eigenvalue $c_0 = 0$ of the operator \hat{C} enters. As fully discussed in Appendix C3, one finds a linear divergence as $k \rightarrow 0$ in c'_1 (whatever the order of the limits $k_0 \rightarrow 0$ and $k \rightarrow 0$), whose origin is the zero eigenvalue of \hat{C} (see Eqs.(C22,C28)).

The contribution from $\Phi'_{(3g)}$ vanishes for l odd, as the entering matrix elements vanish for $k_0 = 0$. For l even, only the sector $|m| = 1$ of $(v \cdot K + i\hat{C})^{-1}$ enters, the contribution is infrared finite (see Appendix C2).

B. The colour conductivity

The properties of this quantity are first summarized for the case when gT has been integrated out. In Sec.III B, the induced current $j_\mu^{ind}(X)$ has been written in terms of the soft field $E^b(Y)$. One may introduce the conductivity tensor

$$j_\mu^a(X) = \int d_4Y \sigma_{\mu i}^{ab}(X, Y) E_i^b(Y) \quad (5.8)$$

The comparison with Eq.(3.23) in Sec.III B gives

$$\sigma_{\mu i}^{ab}(X, Y) = im_D^2 \langle v_\mu G_{ret}^{ab}(X, Y; A; \mathbf{v}, \mathbf{v}') v_i' \rangle_{v, v'} \quad (5.9)$$

The linearized part is in momentum space

$$\sigma_{\mu i}^{ab}(K)|_{A=0} = im_D^2 \langle v_\mu (v.K + i\hat{C})^{-1} v_i' \rangle_{v, v'} \delta^{ab} \quad (5.10)$$

$$\Pi^{\mu i}(K) = -ik_0 \sigma^{\mu i}(K) \quad (5.11)$$

One introduces the transverse and the longitudinal part of the conductivity [10]

$$\begin{aligned} E^i(K) &= \hat{k}^i E_l + E_t^i \\ \sigma_t &= \frac{i}{k_0} \Pi^t = \frac{i}{2k_0} \mathcal{P}_t^{ij} \Pi^{ij} \\ \sigma_l &= -i \frac{k_0}{k^2} \Pi^l = i \frac{k^i}{k^2} \Pi^{0i} \end{aligned}$$

$$\sigma_t = im_D^2 \frac{1}{2} \langle v_t^i (v.K + i\hat{C})^{-1} v_t^i \rangle \quad (5.12)$$

$$\sigma_l = im_D^2 \frac{1}{k^2} \langle k_0 (v.K + i\hat{C})^{-1} \mathbf{v} \cdot \mathbf{k} \rangle \quad (5.13)$$

With $v.K = k_0 - \mathbf{v} \cdot \mathbf{k}$ and relation (2.16), the expression for σ_l may be written in alternative forms

$$\sigma_l = im_D^2 \frac{1}{k^2} \langle \mathbf{v} \cdot \mathbf{k} (v.K + i\hat{C})^{-1} \mathbf{v} \cdot \mathbf{k} \rangle \quad (5.14)$$

$$\sigma_l = im_D^2 \frac{k_0}{k^2} [k_0 \langle (v.K + i\hat{C})^{-1} \rangle - 1] \quad (5.15)$$

i.e. in terms respectively of the matrix elements $l = m = 1$ and $l = m = 0$ of $(v.K + i\hat{C})^{-1}$

$$\sigma_t = \frac{im_D^2}{3} \Sigma_1(k_0, k) \quad (5.16)$$

$$\sigma_l = im_D^2 \frac{k_0}{k^2} (k_0 \Sigma_0(k_0, k) - 1) \quad (5.17)$$

In Appendix C, Eq.(C49) gives $k_0(k_0 \Sigma_0 - 1)/k^2$ as a compact continued fraction for an easy comparison with Eq.(A5) for Σ_1 .

For $k < \gamma/3$, the expansion in k^2 of the continued fraction converges for all k_0 (see Appendix A) and one obtains

$$\begin{aligned}\sigma_t(k_0, k \ll \gamma) &= \frac{im_D^2}{3} \left[\frac{1}{k_0 + ic_1} + \frac{1}{5} \frac{k^2}{(k_0 + ic_1)(k_0 + ic_2)} + O(k^4) \right] \\ \sigma_l(k_0, k^2 \ll k_0\gamma) &= \frac{im_D^2}{3} \left[\frac{1}{k_0 + ic_1} + \frac{4}{15} \frac{k^2}{(k_0 + ic_1)(k_0 + ic_2)} + O(k^4) \right]\end{aligned}$$

i.e. σ_t and σ_l differ by terms of order $k^2/(k_0 + ic_1)(k_0 + ic_2)$. For σ_l , a different limit is (see (C49))

$$\sigma_l(k_0\gamma \ll k^2) = -im_D^2 \frac{k_0}{k^2}$$

We now examine how the collisions at the scale $g^2T \ln 1/g$ affect the conductivities. As said at the beginning of Sec.V, the solution that includes both types of collisions is

$$\Pi^{ji}(K) = k_0 m_D^2 < v^j (v \cdot K + i\hat{C} + i\hat{C}')^{-1} v^i > \quad (5.18)$$

As a consequence, in the expressions just written for σ_t, σ_l , i.e. Eqs.(5.12, 5.14, 5.15), one just have to substitute $\hat{C} + \hat{C}'$ to \hat{C} , i.e. substitute $c_l + c'_l$ to c_l in every continued fraction, in particular

$$\sigma_t(k_0, k \ll \gamma) = \sigma_l(k_0, k^2 \ll k_0\gamma) = \frac{m_D^2}{3} \frac{i}{k_0 + i(c_1 + c'_1)} \quad (5.19)$$

so that

$$\sigma_t = \sigma_l = \sigma(k_0 \ll \gamma, k < k_0) = \frac{m_D^2}{3} \frac{1}{c_1 + c'_1} \quad (5.20)$$

A Comparison with related work

We now examine how these results are related to the pioneers' work of Arnold and Yaffe [11]. Their interest is in the next-to-leading-log contribution to the colour conductivity. Their approach makes use of effective theories. They are lead to compute the same pair of one-loop diagrams contributing to a self-energy in an effective static theory, in the Coulomb gauge. Their operator $\hat{O}(\mathbf{0})$ is identical to the collision operator $i\hat{C}'$, if in \hat{C}' one makes the static approximation $k_0 = 0$ in the effective vertices, and one performs the integral over k_0 with only the weight of the soft transverse propagator; it is an approximation which is done at the very end in this work, they do it at the start. Their W field propagator

$$\hat{G}_0(\mathbf{k}) = i(v \cdot K + i\hat{C})^{-1}|_{k_0=0}$$

is a real quantity (see the matrix elements $i\Sigma_m(k_0 = 0, k)$ in Eq.(A5)). They are lead to consider the same quantity

$$< v_l \hat{O}(\mathbf{0}) v_l > = i c'_1$$

and they find that the relevant matrix element is

$$< v_l v_i \hat{G}_0(\mathbf{k}) v_j v_l > \mathcal{P}_t^{ij}(k)$$

in agreement with $M_{(4g)}^{(l=1)}(k_0 = 0, k)$ in Eq.(5.5). The expression of this matrix element in terms of those of $\hat{G}_0(\mathbf{k})$ agrees with Eq.(C32) in Appendix C 3. They encounter the infrared linear divergence of c'_1 which disappears upon dimensional regularization. At the soft level (gT integrated out), their conductivity tensor agrees with Eq.(5.14) for σ_l . However, in their approach, once the contribution of the momenta $k \sim g^2T \ln 1/g$ is included, the effective conductivity is not merely obtained by the substitution of $c_1 + c'_1 = c_1(1 + c'_1/c_1)$ to c_1 in the expression for σ_l , another term is added arising from the small momentum expansion of their self-energy (see a detailed comparison after Eq.(C39) in App.C 3).

VI. CONCLUSION

To the near-forward collision of two collective colour excitations of the thermal gluons $p \sim T$ is associated a collision operator \hat{C} when the gluon exchanged during the collision has $k \sim gT$. This operator has an infinite number of eigenvalues c_l that may be interpreted as the multipole moments of a rate. The $l = 0$ eigenvalue is zero, a cancellation between a damping term and another term, the other c_l are dominated by the damping term, whose scale is $g^2T \ln 1/g$. When inserted into the transport equation that describes the evolution at some space-time scale x of the collective excitations, the collision term accounts for the effective damping of the excitations due to the integration over smaller scales.

This work has presented two results. The first result is the explicit form of the effective n -gluon amplitudes when the scale T and gT are integrated out. These amplitudes exhibit remarkable properties.

The picture that emerges follows. The central role is played by the collective colour excitations. The soft gluons of momentum $k \ll gT$ are emitted by the collective excitations that occur at the scale $1/k$. These excitations are associated with the transport equation, not with the structures seen in the resummed gluon propagator (such as quasiparticle poles). The effective amplitudes are tree-like and they obey tree-like Ward identities. The propagator along the tree, the one of the collective excitation, propagates an infinite tower of damping rates and one undamped mode. All the rates show an infrared log divergence. The presence of the undamped mode (the eigenvalue $c_0 = 0$) turns out to be essential i) for the Ward identities to be satisfied, ii) for the strong similitude with the HTL amplitudes.

The second result is, a new collision operator \hat{C}' that allows to take into account the collisions with soft gluon exchange $k \sim g^2T \ln 1/g$ has been found by means of a perturbative approach, i.e. by computing the one-soft-loop diagrams (with loop momentum $k \sim g^2T \ln 1/g$) which enter the polarization tensor. The operator \hat{C}' shares many features of the operator \hat{C} : a zero mode, an infinite number of eigenvalues c'_l which are expressed in terms of the c_l .

For a transverse gluon exchange, the landscape in the infrared has changed. The logarithmic divergence that was entering the damping rates c_l has disappeared. The eigenvalues c'_l are finite and of order g^2T (with some dependence on $\ln 1/g$). There is one exception, the $l = 1$ eigenvalue exhibits a linear infrared divergence, which is linked to the zero mode of the operator \hat{C} (as a result of angular momentum combination).

The properties of \hat{C} and \hat{C}' are so similar that it is easy to take into account the collisions with $k \sim g^2T \ln 1/g$. One just has to substitute $\hat{C} + \hat{C}'$ to \hat{C} in the transport equation, i.e.

$c_l + c'_l$ to c_l in its solution. All what has just been said for the n -gluon amplitudes where \hat{C} enters (T and gT integrated out) are valid for the softer amplitudes where $\hat{C} + \hat{C}'$ enters; they are tree-like and satisfy tree-like Ward identities. The quantities $c_l + c'_l$ are infrared finite except for the $l = 1$ case. This $l = 1$ eigenvalue dominates the polarization tensor at momentum $q \ll g^2 T \ln 1/g$. One possible interpretation of this linear infrared divergence is that it is the signal of new physics occurring at the scale $(g^2 T)^{-1}$.

The collision operator \hat{C} is gauge independent, in contrast the longitudinal gluon exchange in the operator \hat{C}' is very likely gauge dependent.

It remains to be seen whether integrating out the scale $g^2 T \ln 1/g$ only amounts to take into account the collisions through \hat{C}' .

Also, this work sheds a new light on what kind of process is being summed by means of the operator \hat{C} or \hat{C}' in the polarization tensor [11,10]. The soft gluon polarization tensor may be written as an infinite series in powers of \hat{C} (at scale $q \ll gT$) or of \hat{C}' (at scale $q \ll g^2 T \ln 1/g$). \hat{C} and \hat{C}' are obtained from a truncation of part of the effective vertices that enter the one-soft-loop diagrams. The truncation amounts to cut-off one initial propagator of the collective excitation. One term entering the collision operator has been shown to come from a self-energy diagram involving a collective excitation and a soft gluon (in fact a damping rate); the terms in the series in powers of that term have alternating factors, one collective excitation's propagator, one self-energy factor, one propagator ..., i.e. the series is just the resummation of the self-energy of the collective excitation. For the collision's other term, the alternating factors are one collective excitation's propagator, two soft gluons, one propagator Since the exchanged gluons have $k_0 < k$, this is a t channel picture; in the crossed channel, one collective excitation scatters and dies out, it is replaced by another collective excitation at a different space-time point, this transition rate involves a two-gluon exchange. This picture is scale invariant. When one goes from \hat{C} to \hat{C}' the change is in the collective excitation's propagator, a straight-line propagation for \hat{C} , fluctuations in the direction \mathbf{v} for \hat{C}' as a result of the processes included in \hat{C} .

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APPENDIX A: THE ANALYTIC PROPERTIES OF Π IN THE COMPLEX K_0 PLANE

1. An explicit expression for Π^t and Π^l

Arnold and Yaffe [11] have written down the characteristics of the operator $[k_0 - \mathbf{v} \cdot \mathbf{k} + iC(\mathbf{v}, \mathbf{v})]^{-1}$ for the case $k_0 = 0$. The extension to the case $k_0 \neq 0$ is straightforward.

$C(\mathbf{v}, \mathbf{v}')$ commutes with the rotations in \mathbf{v} space, its eigenvectors are the spherical harmonics $\sqrt{4\pi} Y_l^m(\mathbf{v}) = |lm\rangle$ with the measure $d\Omega_{\mathbf{v}}/4\pi$

$$\hat{C} |lm\rangle = c_l |lm\rangle = (\gamma - \delta_l) |lm\rangle \quad (\text{A1})$$

$$c_l = \langle C(\mathbf{v}, \mathbf{v}') P_l(\mathbf{v}, \mathbf{v}') \rangle_{vv'} \quad c_0 = 0, \quad c_l > 0 \quad \text{for } l > 0 \quad (\text{A2})$$

$C(\mathbf{v}, \mathbf{v}')$ is given in (2.4).

If one chooses $\hat{\mathbf{k}}$ as the z axis

$$\mathbf{v} \cdot \mathbf{k} |l m\rangle = k (b_l^{(m)} |l+1 m\rangle + b_{l-1}^{(m)} |l-1 m\rangle) \quad (\text{A3})$$

$$b_l^{(m)} = \sqrt{\frac{(l+1)^2 - m^2}{4(l+1)^2 - 1}} \quad (\text{A4})$$

For fixed m , the matrix $[k_0 - \mathbf{v} \cdot \mathbf{k} + iC(\mathbf{v}, \mathbf{v}')]^{-1}$ is a tri-diagonal matrix whose inverse is known. In particular, the element

$$\Sigma_m = \langle l = m | (k_0 - \mathbf{v} \cdot \mathbf{k} + iC(\mathbf{v}, \mathbf{v}'))^{-1} | l = m \rangle$$

is the continued fraction

$$\Sigma_m(k_0, k) = \frac{1}{k_0 + ic_m} \frac{1}{1 - b_m^{(m)2} \frac{\rho_m^2}{1 - b_{m+1}^{(m)2} \frac{\rho_{m+1}^2}{1 - b_{m+2}^{(m)2} \frac{\rho_{m+2}^2}{1 - \dots}}}} \quad (\text{A5})$$

where

$$\rho_m^2 = \frac{k^2}{(k_0 + ic_m)(k_0 + ic_{m+1})} \quad (\text{A6})$$

the transverse part of the self-energy is

$$\Pi_R^t(k_0, k) = \frac{m_D^2}{3} k_0 \Sigma_1(k_0, k) \quad (\text{A7})$$

since from (2.25), (4.6)

$$\Pi_R^t(k_0, k) = m_D^2 k_0 \langle v^i (v \cdot K + i\hat{C})^{-1} v'^j \rangle_{vv'} (\delta_{ij} - \hat{k}_i \hat{k}_j) \frac{1}{2} \quad (\text{A8})$$

and

$$\frac{1}{4\pi} \mathbf{v}_t \cdot \mathbf{v}'_t = \frac{1}{3} [Y_1^1(\mathbf{v}) Y_1^{1*}(\mathbf{v}') + Y_1^{-1}(\mathbf{v}) Y_1^{-1*}(\mathbf{v}')] \quad (\text{A9})$$

Similarly, from (2.25)

$$\Pi_R^l = -\Pi_{00} = -m_D^2 [k_0 \Sigma_0(k_0, k) - 1] \quad (\text{A10})$$

If one restricts oneself to transverse gluon exchange, to leading order, the eigenvalues of the collision operator, defined in (A1), have the properties

$$\delta_{2l+1} = 0, \quad 0 < \delta_{2l} \leq \delta_2 = \frac{5}{8}\gamma \quad \delta_{2l} \rightarrow \frac{2\gamma}{l} \quad \text{as } l \rightarrow \infty \quad (\text{A11})$$

so that the upper bound $\bar{\delta}$ for δ_l is δ_2 . In the following, the discussion is restricted to Π^t . The extension to any Σ_m is immediate.

2. The singularity-free region in the complex k_0 plane

i) The Legendre Functions

If one sets $\gamma \neq 0$ and $\delta_l = 0$, the continued fraction (A5) is a function of $\rho^2 = (k/(k_0 + i\gamma))^2$, its value is well known from the case $\gamma = 0$

$$\Pi_R^t = \frac{m_D^2}{3} \frac{k_0}{k} [Q_0(\frac{k_0 + i\gamma}{k}) - Q_2(\frac{k_0 + i\gamma}{k})] \quad (\text{A12})$$

The analytic properties of $Q_l(z)$ in the complex z plane are seen from

$$Q_l(z) = \frac{1}{2} \int_{-1}^1 dx \frac{P_l(x)}{z - x}$$

$Q_l(z)$ has a logarithmic singularity at $z + 1$ and $z = -1$, and for $\|z\| > 1$, $Q_l(z)$ has a series expansion

$$Q_l(z) = \sum_{n \geq l} a_{ln} z^{-n} \quad \text{with} \quad a_{ln} \geq 0$$

For $\|z\| > 1$ this series is absolutely convergent i.e. $\sum_n a_{ln} \|z\|^{-n}$ converges.

ii) The case $\delta_l \neq 0$

The continued fraction (A5) can be expanded in powers of k^2 and the domain of convergence of this expansion delimited. In this expansion there appear products of

$$\frac{k^2}{(k_0 + i(\gamma - \delta_m))(k_0 + i(\gamma - \delta_{m+1}))}$$

Since all numerical coefficients of the expansion in k^2 are positive, the series has as an upperbound the series of the modulus of its terms. Then one just needs a lower bound for

$$\|k_0 + i(\gamma - \delta_m)\|^2 = (\text{Re}k_0)^2 + (\text{Im}k_0 + \gamma - \delta_m)^2$$

If $\bar{\delta}$ is the upper bound of the δ_m ($\bar{\delta} < 2\gamma/3$), the lower bounds are

$$\begin{aligned} \text{Im}k_0 + \gamma - \bar{\delta} > 0 &\rightarrow \|k_0 + i(\gamma - \delta_m)\| > \|k_0 + i(\gamma - \bar{\delta})\| \\ \text{Im}k_0 + \gamma < 0 &\rightarrow \|k_0 + i(\gamma - \delta_m)\| > \|k_0 + i\gamma\| \\ -\gamma < \text{Im}k_0 < -(\gamma - \bar{\delta}) &\rightarrow \|k_0 + i(\gamma - \delta_m)\| > |\text{Re}k_0| \end{aligned}$$

For the region $\text{Im}k_0 + \gamma - \bar{\delta} > 0$, the expansion in k^2 of the continued fraction is certainly convergent in the domain $\|k/(k_0 + i(\gamma - \bar{\delta}))\| < 1$ as it is the domain of absolute convergence of the expansion in k^2 of $Q_l((k_0 + i(\gamma - \bar{\delta}))/k)$, i.e., in the complex k_0 plane, the expansion is convergent out of a half disk of radius k centered at $k_0 = -i(\gamma - \bar{\delta})$. Similarly, for the region $\text{Im}k_0 < -\gamma$, it is certainly convergent out of a half disk centered at $k_0 = -i\gamma$. And for $-\gamma < \text{Im}k_0 < -(\gamma - \bar{\delta})$, it is convergent for $|\text{Re}k_0| > k$.

To summarize, the expansion of Π_R^t in powers of k^2 is certainly convergent in the complex k_0 plane out of a domain made of two half disks and a rectangle (see Fig.3). In particular, for $\text{Im}k_0 \geq 0$ the expansion is certainly convergent for

$$(\text{Re}k_0)^2 + (\text{Im}k_0 + \gamma - \bar{\delta})^2 > k^2 \quad \text{with} \quad \bar{\delta} = \sup_m \delta_m < 2\gamma/3 \quad (\text{A13})$$

3. The Imaginary parts of Π_R^t

Π_R^t has two types of imaginary parts along the real k_0 axis.

- i) all along the axis, Π_R^t gets an imaginary part from the eigenvalues c_m of the collision operator, i.e. from the damping of the color excitations arising from collisions at the scale $(gT)^{-1}$.
- ii) out of the domain of convergence of the expansion in k^2 , Π_R^t gets another imaginary part similar to the $i\pi$ term of $Q_{2l}(z)$. For example, for $Q_0(z)$ in the complex z plane,

$$\begin{aligned} \|z\| > 1 \quad Q_0(z) &= \frac{1}{2} \ln\left(\frac{z+1}{z-1}\right) \\ \|z\| < 1 \quad Q_0(z) &= -i\frac{\pi}{2} + \frac{1}{2} \ln\left(\frac{1+z}{1-z}\right) \quad \text{above the cut } z = -1 \text{ to } z = 1 \end{aligned}$$

From the shape of the domain of convergence of the expansion in k^2 (see Fig. 3), this imaginary part may only appear in the region $|\text{Re}k_0| < k$. It may be interpreted as a Landau-type effect for the W field, i.e. the propagating fluctuating W field absorbs (emits) a soft gluon $k \sim g^2 T \ln 1/g$ from the plasma.

From the inequality (A13) one sees that for $k < (\gamma - \bar{\delta})$, the expansion in powers of k^2 is convergent all along the k_0 axis, i.e. the half disk on Fig. 3 don't intersect the real axis. As a consequence, Π^t has no Landau-type imaginary part for $k < (\gamma - \bar{\delta}) \sim \gamma/3$.

This appendix discusses the properties of the retarded amplitude $\Pi_R^t(k_0, k)$. A general property of a retarded propagator is that it is analytic in the upper k_0 plane. So it is for Π_R^t (see Fig.3). The advanced amplitude $\Pi_A^t(k_0, k)$ is just the mirror picture, it is analytic in the lower k_0 plane, its singularities are in the upper plane

$$\Pi_A^t(k_0, k) = [\Pi_R^t(k_0, k)]^*$$

Note that $\Pi_R^t(k_0, k)$ and $\Pi_A^t(k_0, k)$ have no common boundary in the complex k_0 plane. (Π_R^t and Π_A^t are on different Riemann sheets of the Landau cut that arises at the scale $k \sim gT$). When one considers along the real axis $\Pi_R^t(k_0, k) - \Pi_A^t(k_0, k)$ one is subtracting two different functions.

4. Locations of the singularities

We study the tail of the continued fraction Σ_m (see (A5)) .

As $l \rightarrow \infty$, $b_l^{(m)2} \rightarrow 1/4$

i) case $\gamma \neq 0$, $\delta_l = 0$

The tail of the continued fraction Σ_m obeys

$$X = 1 - \frac{\rho^2}{4X} \quad \text{i.e.} \quad X = \frac{1 \pm \sqrt{1 - \rho^2}}{2}$$

Hence one recovers the fact that the continued fraction has a singularity at $\rho^2 = 1 = (k/(k_0 + i\gamma))^2$

ii) case $\gamma \neq 0$, $\delta_l \neq 0$
The same argument gives

$$\rho_l^2 = 1 = \frac{k^2}{[k_0 + i(\gamma - \delta_l)][k_0 + i(\gamma - \delta_{l+1})]} \quad (\text{A14})$$

i.e.

$$k_0 = -i(\gamma - \frac{\delta_l + \delta_{l+1}}{2}) \pm \sqrt{k^2 - (\frac{\delta_l - \delta_{l+1}}{2})^2} \quad (\text{A15})$$

Since $(\delta_l + \delta_{l+1})/2 < \bar{\delta}$ and $(\delta_l - \delta_{l+1})^2 < \bar{\delta}^2$ ($\bar{\delta}$ is the upper bound of δ_l), for $k > \bar{\delta}/2 \sim \gamma/3$ all the singularities are inside the two regions in the complex k_0 plane drawn on Fig.3

$$\sqrt{k^2 - \bar{\delta}^2/4} < |\text{Re}k_0| \leq k \quad , \quad \bar{\delta} > \text{Im}k_0 + \gamma > 0$$

Very likely, a cut links these two regions. As $l \rightarrow \infty$ the singularities tend towards $k_0 = \pm k - i\gamma$ since $\delta_{2l} \rightarrow \gamma/(2l)$ for $l \rightarrow \infty$

APPENDIX B: THE 4-GLUON VERTEX

The 4-gluon vertex $V_{\mu\nu\rho\sigma}^{1234}(P_{1R}, P_{2R}, P_{3A}, P_{4A})$ is given by Eq. (3.44) where \mathcal{R} is the sum of two terms

$$\begin{aligned} \mathcal{R} = & -\frac{1}{2}N(P_3, P_4)(f^{14m}f^{23m} + f^{13m}f^{24m}) \\ < & [v_\mu(v.P_1 + i\hat{C})^{-1}v_\nu + v_\nu(v.P_2 + i\hat{C})^{-1}v_\mu](v.(P_1 + P_2) + i\hat{C})^{-1} \\ & [v_\rho(-v.P_4 + i\hat{C})^{-1}v_\sigma(-p_4^0) + v_\sigma(-v.P_3 + i\hat{C})^{-1}v_\rho(-p_3^0)] >_{all \ v} \\ + < & [v_\mu(v.P_1 + i\hat{C})^{-1}v_\nu + v_\nu(v.P_2 + i\hat{C})^{-1}v_\mu](v.(P_1 + P_2) + i\hat{C})^{-1} \\ & \{f^{13m}f^{24m}(p_1^0 + p_3^0) [N(P_3, P_2 + P_4)v_\rho(-v.P_4 + i\hat{C})^{-1}v_\sigma \\ & - N(P_4, P_1 + P_3)v_\sigma(-v.P_3 + i\hat{C})^{-1}v_\rho >] \\ & + f^{14m}f^{23m} [3 \leftrightarrow 4] \} \end{aligned} \quad (\text{B1})$$

where $[3 \leftrightarrow 4]$ means a term obtained from the factor multiplying $f^{13m}f^{24m}$ by the exchange of all indices of 3 and 4. The first term in (B1) is the analogue of the first term in (3.44) with a colour factor symmetric in 1 and 2 rather than antisymmetric. In both terms of \mathcal{R} , X_1^0, X_2^0 are later times, X_3^0, X_4^0 earlier ones, both are essential for the Ward identities to be satisfied. The symmetries $1 \leftrightarrow 2$ and $3 \leftrightarrow 4$ are explicit in (3.44) and in (B1).

The Ward identity relating $p_1^\mu V_{\mu\nu\rho\sigma}^{1234}$ to 3-point vertices is written in Eq.(3.43) where a vertex of type RRA is related to the one of type ARR (as given in (3.34)) as follows

$$V_{\nu\rho\sigma}^{2m4}(P_{2R}, P_{3R}, P_{4A}) = [V_{\sigma\nu\rho}^{42m}(P_{4R}, P_{2A}, P_{3A})]^+ \quad (\text{B2})$$

where the $+$ operation reverses the string of operators, changes $i\hat{C}$ into $-i\hat{C}$ and reverses the colour order. This is consistent with Eqs.(3.1,3.2) since if^{42m} is unchanged in the $+$ operation. The Ward identity for a leg of type A is

$$\begin{aligned}
ip_4^\sigma V_{\mu\nu\rho\sigma}^{1234}(P_{1R}, P_{2R}, P_{3A}, P_{4A}) &= f^{43m} V_{\mu\nu\rho}^{12m}(P_{1R}, P_{2R}, (P_3 + P_4)_A) \\
&+ f^{42m} \left[\frac{N(P_3, P_2 + P_4)}{N(P_3, P_4)} V_{\mu\nu\rho}^{1m3}(P_{1R}, (P_2 + P_4)_A, P_{3A}) \right. \\
&+ \left. \frac{N(P_4, P_1 + P_3)}{N(P_3, P_4)} V_{\mu\nu\rho}^{1m3}(P_{1R}, (P_2 + P_4)_R, P_{3A}) \right] \\
&+ f^{41m} [1 \leftrightarrow 2]
\end{aligned} \tag{B3}$$

where the 3-point vertices are those of Eq.(3.34) and Eq.(B2).

APPENDIX C: THE EIGENVALUES OF THE COLLISION OPERATOR \hat{C}'

1. The eigenvalues as an integral

$\hat{C}'(\mathbf{v}, \mathbf{v}')$ commutes with the rotations in \mathbf{v} space, its eigenvalues c'_l depend on the only available quantity $\mathbf{v} \cdot \mathbf{v}'$

$$c'_l = \langle \hat{C}'(\mathbf{v}, \mathbf{v}') P_l(\mathbf{v} \cdot \mathbf{v}') \rangle_{\mathbf{v}, \mathbf{v}'} \tag{C1}$$

where P_l is the Legendre polynomial. In this appendix an analytic expression of c'_l is given for the case of the dominant transverse gluon exchange in terms of the matrix elements of the operator $(v \cdot K + i\hat{C})^{-1}$

$$c'_l = m_D^2 \frac{g^2 NT}{2} \int \frac{d_4 k}{(2\pi)^4} |\Delta^t(K_R)|^2 [N_{(3g)}^{(l)}(K) + N_{(4g)}^{(l)}(K)] \tag{C2}$$

from Eqs (4.30,4.31,4.33) and (4.58,4.60), with

$$\begin{aligned}
N_{(3g)}^{(l)}(K) &= \langle P_l(\mathbf{v}_1 \cdot \mathbf{v}_2) \rangle_{\mathbf{v}_1, \mathbf{v}_2} = \langle v_{1i}^t (v_1 \cdot K + i\hat{C})^{-1} v_{1j}^t - v_{1j}^t (v_1 \cdot K - i\hat{C})^{-1} v_{1i}^t \rangle_{\mathbf{v}_1} \\
&< \langle v_{2i}^t (v_2 \cdot K + i\hat{C})^{-1} v_{2j}^t - v_{2j}^t (v_2 \cdot K - i\hat{C})^{-1} v_{2i}^t \rangle_{\mathbf{v}_2} \rangle_{\mathbf{v}_1, \mathbf{v}_2}
\end{aligned} \tag{C3}$$

$$N_{(4g)}^{(l)}(K) = -\left(\frac{2i\text{Im}\Pi^t}{k_0 m_D^2}\right) \langle P_l(\mathbf{v} \cdot \mathbf{v}') v_i^t [(v \cdot K + i\hat{C})^{-1} - (v \cdot K - i\hat{C})^{-1}] v_i^t \rangle_{\mathbf{v}, \mathbf{v}'} \tag{C4}$$

where $v_i^t = v_i - \hat{k}_i \mathbf{v} \cdot \hat{\mathbf{k}}$, and the integration range on k is limited to $k \sim g^2 NT \ln 1/g$, i.e. $\mu_3 < k < \mu_2$ where μ_2 and μ_3 have been defined in Eq.(2.43). In this range

$$\Pi^t(k_0, k) = \frac{k_0 m_D^2}{3} \Sigma_1(k_0, k) \tag{C5}$$

where $\Sigma_1(k_0, k)$ is a matrix element of $(v \cdot K + i\hat{C})^{-1}$ given in Eq.(A5) whose scale is $\gamma \sim g^2 NT \ln 1/g$ for k_0 and k . Then

$$|\Delta^t(K_R)|^2 = \frac{1}{(k_0^2 - k^2 + \text{Re}\Pi^t)^2 + (\text{Im}\Pi^t)^2} \approx \frac{1}{k^4 + k_0^2 m_D^4 (\text{Im}\Sigma_1/3)^2} \tag{C6}$$

In the space (k_0, k) , $|\Delta^t|^2$ put a strong weight upon the region

$$k_0 \frac{m_D^2}{3} \text{Im}\Sigma_1(k_0, k) \leq k^2 \quad (\text{C7})$$

As $m_D^2 \Sigma_1 \sim m_D^2/\gamma \sim T/\ln 1/g$ the weight is on the domain

$$k_0 \leq \frac{k^2}{T} \ln 1/g \quad (\text{C8})$$

The matrix elements of $(v.K + i\hat{C})^{-1}$ all have the same scale γ for both k_0 and k (see Eq.(A5) and see Sec.C 4 of this Appendix) except for a few that depend on the zero eigenvalue of \hat{C} . For those, one may substitute $N^{(l)}(k_0 = 0, k)$ to $N^{(l)}(k_0, k)$ in the integrant of (C2) and perform the integration over k_0 (see Eq. (4.44)) with the result

$$c'_l \approx \frac{g^2 NT}{2} \int_{\mu_3}^{\mu_2} \frac{dk}{4\pi^2} \frac{3}{|\text{Im}\Sigma_1(k_0 = 0, k)|} [N_{(3g)}^{(l)}(k_0 = 0, k) + N_{(4g)}^{(l)}(k_0 = 0, k)] \quad (\text{C9})$$

The integral will have no infrared divergence if $N^{(l)}(k_0 = 0, k)$ is finite as $k \rightarrow 0$ since $\text{Im}\Sigma_1(k_0 = 0, k) \rightarrow -1/\gamma$ as $k \rightarrow 0$. The exceptional cases will be discussed later on.

In the following, $N_{(3g)}^{(l)}(k_0, k)$ and $N_{(4g)}^{(l)}(k_0, k)$ are expressed in terms of matrix elements of $(v.K + i\hat{C})^{-1}$ whose explicit expressions are given in Sec.C 4. Useful properties are:

- The matrix elements of $(v.K + i\hat{C})^{-1}$ are divided into subspaces where $l_z = m = \mathbf{l} \cdot \hat{\mathbf{k}}$ is fixed and $l = m, m+1, m+2, \dots$ (see Appendix A)
- Matrix elements for $l_z = m$ and $l_z = -m$ are equal.
- For $k_0 = 0$ and all k , all the relevant matrix elements are imaginary numbers.

One will write

$$G = (v.K + i\hat{C})^{-1} \quad \text{and} \quad G^+ = (v.K - i\hat{C})^{-1} \quad (\text{C10})$$

$\hat{\mathbf{k}}$ is taken as the z axis, and in $N^{(l)}(k_0, k)$ one writes

$$P_l(\mathbf{v} \cdot \mathbf{v}') = \frac{4\pi}{2l+1} \sum_{m=-l}^{m=l} Y_l^{m*}(\mathbf{v}) Y_l^m(\mathbf{v}') \quad (\text{C11})$$

2. The 3-gluon vertices diagram's contribution

$N_{(3g)}^{(l)}(K)$ is given by Eq.(C3) and (C11) is used. Only the subspace $|m| = 1$ of $(v.K + i\hat{C})^{-1}$ enters. Indeed, v_{1i}^t is an element with $|m| = 1$, $(v.K + i\hat{C})^{-1}$ acts in this subspace, v_{1i}^t and $Y_l^m(\mathbf{v}_1)$ are combined with the use of Clebsh-Gordan coefficients and the \mathbf{v}_1 integration project them into the subspace. Writing for compactness

$$\langle l' \ m = 1 \mid G - G^+ \mid 1 \ m = 1 \rangle = \langle l' \mid \Delta G \mid 1 \rangle \quad (\text{C12})$$

the result is

$$\begin{aligned}
N_{(3g)}^{(l)} = & \frac{1}{3(2l+1)^2} \{ [\left(\frac{l(l-1)}{2l-1} \right)^{1/2} \langle l-1 | \Delta G | 1 \rangle - \left(\frac{(l+1)(l+2)}{2l+3} \right)^{1/2} \langle l+1 | \Delta G | 1 \rangle]^2 \\
& + [\left(\frac{(l+1)(l+2)}{2l-1} \right)^{1/2} \langle l-1 | \Delta G | 1 \rangle - \left(\frac{l(l-1)}{2l+3} \right)^{1/2} \langle l+1 | \Delta G | 1 \rangle]^2 (1-\delta_{l0})(1-\delta_{l1}) \}
\end{aligned} \tag{C13}$$

The first term comes from the $m = 0$ term in the sum (C11), the second from the $|m| = 2$ term.

As the matrix elements have the property (see Eq.(C50))

$$\langle 1+n | G^+(k_0, k) | 1 \rangle = (-1)^{n-1} \langle 1+n | G(-k_0, k) | 1 \rangle \tag{C14}$$

$\langle l \pm 1 | \Delta G | 1 \rangle$ is an even function of k_0 for l even, an odd function of k_0 for l odd. $N_{(3g)}^{(l)}(k_0 = 0, k) = 0$ for l odd, i.e. with the approximation of (C9) the contribution of $N_{(3g)}^{(l)}$ to c'_l vanishes for l odd.

The matrix elements $\langle l \pm 1 | G | 1 \rangle$ may be expressed in terms of $\langle 1 | G | 1 \rangle = \Sigma_1$ (See Eq.(C41)). Explicitely

$$N_{(3g)}^{(l=0)}(k_0, k) = -\frac{2}{9} [2 \text{Im} \Sigma_1]^2 \tag{C15}$$

$$N_{(3g)}^{(l=1)}(k_0, k) = \frac{1}{5^2} \frac{2}{9} k^2 [\frac{\Sigma_1}{(k_0 + ic_2)X_2} - \text{c.c.}]^2 \tag{C16}$$

$$\begin{aligned}
N_{(3g)}^{(l=2)}(k_0, k) = & \frac{1}{5^2} \frac{2}{9} \{ [\Sigma_1 (1 - \frac{3.4}{5.7} \frac{k^2}{(k_0 + ic_2)(k_0 + ic_3)} \frac{1}{X_2 X_3}) - \text{c.c.}]^2 \\
& + 6 [\Sigma_1 (1 - \frac{2}{5.7} \frac{k^2}{(k_0 + ic_2)(k_0 + ic_3)} \frac{1}{X_2 X_3}) - \text{c.c.}]^2 \}
\end{aligned} \tag{C17}$$

where c.c. means complex conjugate, $\Sigma_1(k_0, k)$ is given in (A5), X_2, X_3 are defined in term of Σ_1 in (C38,C39), and c_l is the l eigenvalue of the operator \hat{C} .

For $k \ll \gamma$, $\Sigma_1 \approx (k_0 + ic_1)^{-1}$ and from Eq.(C41) $\langle l+1 | G | 1 \rangle$ behaves as k^l , $\langle l-1 | G | 1 \rangle$ as k^{l-2} . One concludes that the contribution of $N_{(3g)}^{(l)}$ to the integral (C9) is infrared finite for all l .

3. The 4-gluon vertex diagram's contribution

$N_{(4g)}^{(l)}(K)$ is written Eq.(C4) and (C11) is used. $v_i^t Y_l^m(\mathbf{v})$ has components in the $m+1$ and $m-1$ subspaces of $G = (v.K + i\hat{C})^{-1}$. The result is written as a sum over the different subspaces M of G , the two first terms correspond to $M = l+1$ and $M = l$, $\Delta G = G - G^+$.

$$\begin{aligned}
N_{(4g)}^{(l)} = & -i \frac{2}{3} \text{Im} \Sigma_1 \frac{2}{2l+1} \{ \\
& \frac{l+1}{2l+3} \langle l+1 \ l+1 | \Delta G | l+1 \ l+1 \rangle + \frac{l}{2l+3} \langle l+1 \ l | \Delta G | l+1 \ l \rangle \\
& + \sum_{M=0}^{M=l-1} (1 - \frac{1}{2} \delta_{M0}) [\frac{l(l+1) + M^2}{2l+1} (\frac{1}{2l+3} \langle l+1 \ M | \Delta G | l+1 \ M \rangle \\
& + \frac{1}{2l-1} \langle l-1 \ M | \Delta G | l-1 \ M \rangle) - b_l^{(M)} b_{l-1}^{(M)} 2 \langle l+1 \ M | \Delta G | l-1 \ M \rangle (1 - \delta_{l0})] \}
\end{aligned} \tag{C18}$$

with $b_l^{(M)}$ given in (A4). With the property of G (see Eq.(C50))

$$\langle l' + n \ M | G^+(k_0, k) | l' \ M \rangle = (-1)^{n+1} \langle l' + n \ M | G(-k_0, k) | l' \ M \rangle \tag{C19}$$

each matrix element of (C18) is an even function of k_0 . The matrix elements in a given subspace M may be written in terms of the lowest one

$$\Sigma_M = \langle l = M \ M | G | l = M \ M \rangle \tag{C20}$$

For example

$$N_{(4g)}^{(l=0)} = \frac{2}{9} [2 \text{Im} \Sigma_1]^2 \tag{C21}$$

the contribution of $N_{(4g)}^{(l=0)}$ and $N_{(3g)}^{(l=0)}$ cancel each other and $c'_0 = 0$ as it was shown in Sec.IV D by another method.

$$\begin{aligned}
N_{(4g)}^{(l=1)} = & -i \frac{2}{3} \text{Im} \Sigma_1 \frac{2}{3} \{ \frac{2}{5} \langle 2 \ 2 | \Delta G | 2 \ 2 \rangle + \frac{1}{5} \langle 2 \ 1 | \Delta G | 2 \ 1 \rangle \\
& + \frac{1}{3} [\langle 0 \ 0 | \Delta G | 0 \ 0 \rangle + \frac{1}{5} \langle 2 \ 0 | \Delta G | 2 \ 0 \rangle - \frac{2}{\sqrt{5}} \langle 2 \ 0 | \Delta G | 0 \ 0 \rangle] \}
\end{aligned} \tag{C22}$$

With the use of the relations between the matrix elements of G , it may be written in terms of $\Sigma_2, \Sigma_1, \Sigma_0$ associated respectively with the subspaces $M = 2, 1, 0$. For example

$$\langle 2 \ 1 | \Delta G | 2 \ 1 \rangle = 5 \frac{d_1}{k^2} (d_1 \Sigma_1 - 1) = \frac{d_1}{d_2} \frac{\Sigma_1}{X_2^{(1)}} \tag{C23}$$

where $d_l = k_0 + ic_l$ and the $X_l^{(m)}$ are expressed in terms of Σ_m in (C39,C38).

$$\begin{aligned}
N_{(4g)}^{(l=1)}(k_0, k) = & -i \frac{2}{3} \text{Im} \Sigma_1 \frac{2}{3} \{ \frac{2}{5} (\Sigma_2 - \text{c.c.}) + \frac{1}{5} [\frac{k_0 + ic_1}{k_0 + ic_2} \frac{1}{X_2^{(1)}} \Sigma_1 - \text{c.c.}] \\
& + \frac{1}{3} [\Sigma_0 (1 - \frac{1}{3} \frac{k^2}{(k_0 + ic_1)(k_0 + ic_2)} \frac{1}{X_1^{(0)} X_2^{(0)}} + \frac{1}{5} \frac{k_0}{k_0 + ic_2} \frac{1}{X_1^{(0)} X_2^{(0)}}) - \text{c.c.}]
\end{aligned} \tag{C24}$$

where the c_l are the eigenvalues of \hat{C} , and the $X_l^{(m)}$ are expressed in terms of Σ_m in (C38,C39).

The infrared sector

For $k \ll \gamma$, the diagonal matrix elements of G do not depend on the subspace M

$$\langle l' M | G | l M \rangle = \langle l' M | (v.K + i\hat{C})^{-1} | l M \rangle \approx (k_0 + ic_l)^{-1} \delta_{ll'} \quad (\text{C25})$$

With $\sum_1^{l-1} M^2 = l(l-1)(l-1/2)/3$, one can check that the full expression (C18) reduces to

$$N_{(4g)}^{(l)}(k_0, k \ll \gamma) = -i \frac{2}{3} \text{Im}\Sigma_1 \frac{2}{2l+1} \frac{1}{3} \left[\frac{l+1}{k_0 + ic_{l+1}} + \frac{l}{k_0 + ic_{l-1}} - \text{c.c.} \right] \quad (\text{C26})$$

a form immediately obtained from the initial expression of $N_{(4g)}^{(l)}$, Eq.(C4), if one writes

$$\mathbf{v}_t \cdot \mathbf{v}'_t P_l(\mathbf{v} \cdot \mathbf{v}') = \frac{2}{3} \mathbf{v} \cdot \mathbf{v}' P_l(\mathbf{v} \cdot \mathbf{v}') = \frac{2}{3} \frac{1}{2l+1} [(l+1)P_{l+1} + lP_{l-1}] \quad (\text{C27})$$

Because of the eigenvalue $c_0 = 0$, care has to be given to the subspace $M = 0$. As shown in Sec.C4 of this appendix, the limits $k \rightarrow 0$ and $k_0 \rightarrow 0$ commute for all $\langle l' 0 | G | l 0 \rangle$ with l and $l' \geq 2$. One concludes that the contribution of $N_{(4g)}^{(l)}$ to the eigenvalue c'_l is infrared finite for $l > 2$. For the case $l = 2$, the elements $\langle 1 0 | G | l' 0 \rangle$ enter in (C18), the limits $k \rightarrow 0$ and $k_0 \rightarrow 0$ do not commute, these elements vanish as $k_0 \rightarrow 0$, k fixed (see Eq.(C47)). There is no infrared divergence in $N_{(4g)}^{(l=2)}$.

The case of c'_1

As the relation (C26) suggests, one is left with the case $l = 1$ whose explicit expression is written in (C22) or (C24). In (C22), the matrix elements $\langle 2 M | \Delta G | 2 M \rangle \rightarrow 2/i c_2$ as $k \rightarrow 0$ irrespective of the order of the limits k_0 and k , they give a finite contribution to c'_1 . The other terms are

$$\begin{aligned} & \frac{1}{3} \left[\langle 0 0 | \Delta G | 0 0 \rangle - \frac{2}{\sqrt{5}} \langle 0 0 | \Delta G | 2 0 \rangle \right] = \\ & \frac{1}{3} \Sigma_0 \left(1 - \frac{k^2}{(k_0 + ic_1)(k_0 + ic_2)} \frac{1}{X_1^{(0)} X_2^{(0)}} \frac{4}{15} \right) - \text{c.c.} \end{aligned} \quad (\text{C28})$$

if one uses (C41) to relate the second matrix element to the first one. With

$$\Sigma_0(k_0 = 0, k) = -\frac{3ic_1}{k^2} X_1^{(0)} \quad (\text{C29})$$

where $X_l^{(0)}(k_0 = 0, k)$ real > 1 from Eqs.(C38,C39), one sees that the integral (C9) has a linear infrared divergence which is solely due to the existence of the matrix element $\langle 0 0 | \Delta G | 0 0 \rangle = 2i \text{Im}\Sigma_0$ in (C28).

Instead of considering $\Sigma(k_0 = 0, k)$ as it enters (C9), one may wish to go back one step ahead and study the relevant domain (k_0, k) in $\int k^2 dk dk_0$ of Eq.(C2). Here

$$\int k^2 dk dk_0 \frac{1}{k^4 + k_0^2 a^2} \frac{k^2}{k^4 + k_0^2 b^2} = \pi \int \frac{dk}{k^2} \frac{1}{a + b} \quad (\text{C30})$$

From $|\Delta^t|^2$ in (C6) one has $a \sim m_D^2/c_1$, and for $\text{Im}\Sigma_0$ one has $b \sim c_1$ since from (C38), for an estimate

$$\text{Im}\Sigma_0 \approx \text{Im}\frac{1}{k_0 - \frac{k^2}{3(k_0 + ic_1)}} = -\frac{1}{3} \frac{c_1 k^2}{(k_0^2 - \frac{k^2}{3})^2 + c_1^2 k_0^2} \approx -3 \frac{c_1 k^2}{k^4 + 9c_1^2 k_0^2} \quad (\text{C31})$$

With $m_D^2 = g^2 NT^2/3 \gg c_1^2 \sim (g^2 NT \ln 1/g)^2$, one has $a \gg b$ in (C30) and one concludes that the linear divergence is indeed given by $\Sigma_0(k_0 = 0, k)$.

If one sets $k_0 = 0$ and one writes $\Sigma_l(k_0 = 0, k) = -i\tilde{\Sigma}_l(k)$, one gets from (C22)

$$N_{(4g)}^{(l)}(k_0 = 0, k) = \tilde{\Sigma}_1 \left[\frac{4}{9} \left[\frac{4}{5} \tilde{\Sigma}_2 + 2 \frac{c_1}{k^2} (1 - c_1 \tilde{\Sigma}_1) + \frac{2}{3} (\tilde{\Sigma}_0 - 3 \frac{c_1}{k^2}) (1 + \frac{5}{4}) + 2 \frac{c_1}{k^2} \right] \right] \quad (\text{C32})$$

As $k \rightarrow 0$, all the terms are finite, except for the term $2c_1/k^2$. In the factor $(1 + 5/4) = 9/4$, 1 comes from the infrared singular element $\langle 0 \ 0 \mid \Delta G \mid 0 \ 0 \rangle$ and $5/4$ from the elements $\langle 2 \ 0 \mid \Delta G \mid 2 \ 0 \rangle$ and $\langle 0 \ 0 \mid \Delta G \mid 2 \ 0 \rangle$. The form (C32) allows an easy comparison with the result of Arnold and Yaffe [11] at the end of their subsec.7 of Sec.II.C. As it has been discussed in Sec.VB, comparing their expressions at the end of their subsecs.7 and 5, one has to substract out the ρ -dependent part which does not come from $\langle v_l v_i G_0(p) v_j v_l \rangle \mathcal{P}_t^{ij}$ but from

$$\frac{2}{3p^2} (4 - 3 \frac{\sigma_p}{\sigma_0}) = \frac{8}{3p^2} - \frac{2\Sigma_1}{p^2}$$

i.e. for the term in $(-\Sigma_1)$, writing in their subsec.7's form $8/3 = 2 + 2/3$, it remains

$$\frac{1}{2} (\Sigma_0 - \frac{3}{\rho^2}) + \frac{2}{3\rho^2} (1 - \Sigma_1) + \frac{4}{15} \Sigma_2$$

which agrees with the convergent term in the bracket in Eq.(C32) when one factors out $1/3$.

4. Matrix elements of $G = (v.K + i\hat{C})^{-1}$

The eigenvectors of $\hat{C}(\mathbf{v}, \mathbf{v}')$ are $|l \ m\rangle = \sqrt{4\pi} Y_l^m(\mathbf{v})$ with the measure $d\Omega_{\mathbf{v}}/4\pi$, its eigenvalues are c_l . It is convenient to choose $\hat{\mathbf{k}}$ as the z axis, then the operator $(k_0 + i\hat{C} - v_z k)$ changes l but does not change $l_z = m$, so does its inverse

$$\langle l' \ m \mid (k_0 + i\hat{C} - v_z k)^{-1} \mid l \ m \rangle = \delta_{mm'} \langle l' \ m \mid (k_0 + i\hat{C} - v_z k)^{-1} \mid l \ m \rangle \quad (\text{C33})$$

In a given subspace m , all matrix elements of G may be written in terms of the lowest one [11]

$$\langle l = m \ m \mid G \mid l = m \ m \rangle = \Sigma_m \quad (\text{C34})$$

because of the recursion relations

$$\langle l' \ m \mid G \ G^{-1} \mid l \ m \rangle = \delta_{ll'} \quad (\text{C35})$$

$$\begin{aligned} \delta_{ll'} &= -kb_{l'} \langle l' + 1 \mid G \mid l \rangle - kb_{l'-1} \langle l' - 1 \mid G \mid l \rangle + d_{l'} \langle l' \mid G \mid l \rangle \\ \delta_{ll'} &= -kb_l \langle l' \mid G \mid l + 1 \rangle - kb_{l-1} \langle l' - 1 \mid G \mid l \rangle + d_l \langle l' \mid G \mid l \rangle \end{aligned} \quad (\text{C36})$$

with the definition

$$\langle l | k_0 + i\hat{C} | l \rangle = k_0 + ic_l = d_l \quad (\text{C37})$$

The $b_l = b_l^{(m)}$ are the matrix elements of v_z defined in (A4), and the reference to the subspace m has been dropped. Note that a consequence of (C36) is, as $k \rightarrow 0$, $\langle l' | G | l \rangle \rightarrow \delta_{ll'}/d_l$ independent of m .

It is convenient to exhibit the threshold factors arising from angular momentum conservation. Σ_m is the continued fraction explicated in Eq.(A5), it may be written

$$\Sigma_m(k_o, k) = \frac{1}{d_m} \frac{1}{1 - \frac{k^2 b_m^2}{d_m d_{m+1}} \frac{1}{X_{m+1}}} \quad (\text{C38})$$

$$X_{m+1} = 1 - \frac{k^2 b_{m+1}^2}{d_{m+1} d_{m+2}} \frac{1}{X_{m+2}} \quad (\text{C39})$$

with $d_m = k_0 + ic_m$ (see (C37)), $b_{m+n} = b_{m+n}^{(m)}$ is the quantity that depends on the subspace m (see (A4))

$$b_0^{(0)^2} = \frac{1}{3} \quad , \quad b_1^{(0)^2} = \frac{4}{15} \quad , \quad b_1^{(1)^2} = \frac{1}{5} \quad (\text{C40})$$

So do the X_m . Then the solution to the recursion relations *in the subspace m* may be written for any $l \geq m$, $n > 0$

$$\begin{aligned} \langle l+n | G | l \rangle &= \langle l | G | l+n \rangle = \\ &= \frac{b_l b_{l+1} \dots b_{l+n-1}}{X_{l+1} X_{l+2} \dots X_{l+n}} \frac{k^n}{d_{l+1} d_{l+2} \dots d_{l+n}} \langle l | G | l \rangle \end{aligned} \quad (\text{C41})$$

and $\langle l | G | l \rangle$ is related as follows to the lowest $l = m$

$$\langle l = m | G | l = m \rangle = \Sigma_m \quad (\text{C42})$$

$$\langle l = m+1 | G | l = m+1 \rangle = \frac{d_m}{d_{m+1} X_{m+1}} \Sigma_m \quad (\text{C43})$$

$$\langle l = m+2 | G | l = m+2 \rangle = \frac{d_m}{d_{m+2} X_{m+1} X_{m+2}} \left(1 - \frac{k^2 b_m^2}{d_m d_{m+1}}\right) \Sigma_m \quad (\text{C44})$$

$$\langle l = m+3 | G | l = m+3 \rangle = \frac{d_m}{d_{m+3} X_{m+1} X_{m+2} X_{m+3}} \left(1 - \frac{k^2 b_m^2}{d_m d_{m+1}} - \frac{k^2 b_{m+1}^2}{d_{m+1} d_{m+2}}\right) \Sigma_m \quad (\text{C45})$$

Limits as k_0 or $k \rightarrow 0$

i) For any subspace $m \neq 0$

- as $k \rightarrow 0$, all $X_m \rightarrow 1$, $\Sigma_m \rightarrow 1/d_m$, one finds $\langle l | G | l \rangle \rightarrow 1/d_l$ independent of the subspace m as expected, and one obtains the correct threshold factor for the non-diagonal matrix elements.

- as $k_0 \rightarrow 0$, $d_m = i c_m$, $X_m > 1$, all $\langle l | G | l \rangle$ are imaginary, and the non-diagonal matrix elements are alternatively real and imaginary.

ii) For the subspace $m = 0$, $d_0 = k_0$ as $c_0 = 0$, and care is needed when taking the limits

- $k_0 \neq 0$ the limit $k \rightarrow 0$ is as in the other subspaces m

- k fixed, $k_0 \rightarrow 0$, $d_0 \rightarrow 0$, $d_m \rightarrow i c_m$, $X_m > 1$. From (C38)

$$\langle 0 0 | G | 0 0 \rangle = \Sigma_0 = -\frac{d_1 X_1}{k^2 b_0^2} = -i \frac{c_1 X_1}{k^2 b_0^2} \quad (\text{C46})$$

For $l \geq 2$, the elements $\langle l 0 | G | l 0 \rangle$ have factors

$$\frac{1}{d_l} \left(d_0 \left(1 - \frac{k^2 b_0^2}{d_0 d_1} \right) + \dots \right) \Sigma_0 \xrightarrow{k_0 \rightarrow 0} \frac{1}{d_l} \left(-\frac{k^2 b_0^2}{d_1} + O(k^4) \right) \Sigma_0$$

so that the limit $k \rightarrow 0$ again gives $1/d_l$. As a consequence, the limits $k \rightarrow 0$ and $k_0 \rightarrow 0$ commute in all matrix elements $\langle l 0 | G | l' 0 \rangle$ with l and $l' \geq 2$.

The exceptions are seen from Eqs.(C41),(C43),(C46). For $k_0 \rightarrow 0$, k fixed,

$$\langle 1 0 | G | l 0 \rangle \rightarrow 0 \quad \text{for } l \geq 1 \quad (\text{C47})$$

$$\langle 0 0 | G | 1 0 \rangle = \frac{b_0 k \Sigma_0}{d_1 X_1} \rightarrow -\frac{1}{k b_0} \quad (\text{C48})$$

and $\langle 0 0 | G | l 0 \rangle$ have the anomalous threshold factor k^{l-2} .

Another quantity to be used is

$$\frac{k_0}{k^2} (k_0 \Sigma_0 - 1) = \frac{1}{3} \frac{k_0}{k_0 d_1 \left(1 - \frac{k^2 b_1^{(0)2}}{d_1 d_2 X_2^{(0)}} \right) - \frac{k^2}{3}} \quad (\text{C49})$$

where $d_1 = k_0 + i c_1$. There are two limits:

$k^2 \ll k_0 |k_0 + i c_1|$, the limit is $1/3 d_1$

$k_0 |k_0 + i c_1| \ll k^2$, the limit is $-k_0/k^2$

Complex conjugation

The matrix elements of $G = (v.K + i\hat{C})^{-1}$ and of $G^+ = (v.K - i\hat{C})^{-1}$ may be related. Writing $d_m^*(k_0) = k_0 - i c_m = -d_m(-k_0)$, one has from Eqs.(C38) to (C42)

$$\begin{aligned} X_m^*(k_0, k) &= X_m(-k_0, k) \\ \Sigma_m^*(k_0, k) &= -\Sigma_m(-k_0, k) \\ \langle l | G^+(k_0, k) | l + n \rangle &= (-1)^{n+1} \langle l | G(-k_0, k) | l + n \rangle \end{aligned} \quad (\text{C50})$$

Limit when \hat{C} is replaced by ϵ

$$\begin{aligned} \langle l' m | (v.K + i\epsilon)^{-1} | l m \rangle = & \quad (C51) \\ (-1)^m \sum_{L=l-l'}^{L=l+l'} \frac{1}{k} Q_L\left(\frac{k_0 + i\epsilon}{k}\right) [(2l+1)(2l'+1)]^{1/2} \langle l l'; 0 0 | L 0 \rangle \langle l l'; m - m | L 0 \rangle \end{aligned}$$

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FIGURES

FIG. 1. (a) (b) (c) are the three possible ways of joining two 3-point vertices in the R/A formalism.

FIG. 2. (a) Contribution to the self-energy of the 4-gluon vertex' diagram. (b) The surviving diagram for a soft loop; the double arrow indicates the ϵ flow.

FIG. 3. Analytic properties of $\Pi_R^t(k_0, k)$. The singularities are inside the rectangles; the expansion in k^2 is convergent out of the domain limited by the dashed lines.

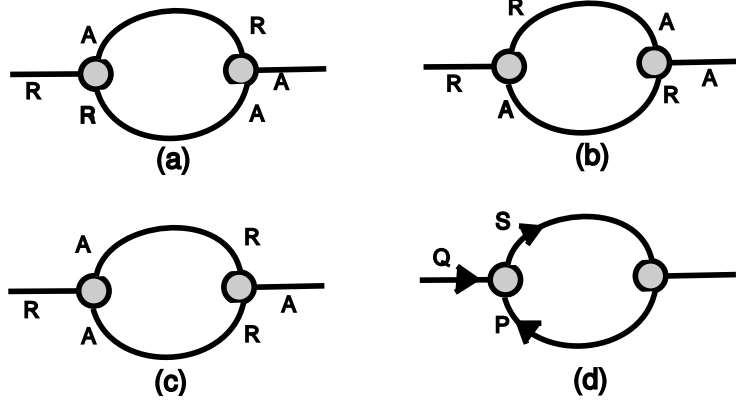


Figure 1: (a) (b) (c) are the three possible ways of joining two 3-point vertices in the R/A formalism.

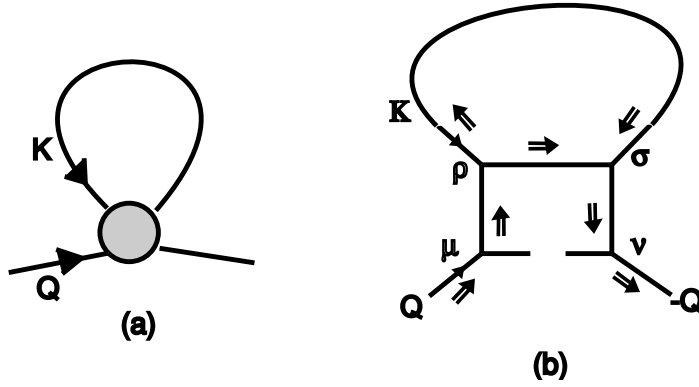


Figure 2: (a) Contribution to the self-energy of the 4-gluon vertex' diagram. (b) The surviving diagram for a soft loop; the double arrow indicates the ϵ flow.

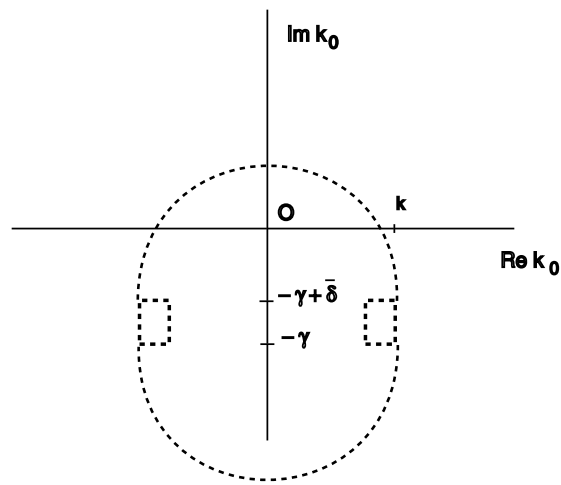


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